Heisenberg Invariant Elliptic Curves

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Contents

1 Introduction

In this report we want to investigate the symmetries of elliptic curves $E \subseteq \mathbb{P}^{n-1}$ of degree $n \geq 3$, over the complex numbers $\mathbb{C}$. Section 2 introduces this notion by viewing the an elliptic curve as the quotient of $\mathbb{C}$ by a lattice, and also the complementary viewpoint. Whilst completely elementary, opens up the discussion for subsequent sections.

After this motivation, in Section 3 we introduce theta functions relative to a lattice, and study some of their properties. In particular, we define theta functions of weight n for some positive integer n with form a vector space of dimension n. The motivation behind this result is that for $n \geq 3$, the basis theta functions define a holomorphic embedding of the elliptic curve (as a complex torus) into $\mathbb{P}^{n-1}$, which by Chow's Theorem defines an algebraic variety.

Section 4 is dedicated to the relations satisfied between the theta functions, as to describe the model of the elliptic curve that is given rise to. We start off with the case when $n = 3$, and find that the embedded elliptic curve takes on the form of a homogeneous cubic polynomial that belongs to the Hesse family. We then generalise this result to all positive $n \geq 4$, and prove that the embedded elliptic curve is described the the set-theoretic intersection of $n(n-3)/2$ linearly independent quadrics, and discuss some interesting cases for low values of n which exhibit some interesting symmetries.

In Section 5 we introduce the Heisenberg group, which lifts the n-translation point action to $\mathbb{P}^{n-1}$, and discuss study the invariant hyperplanes of this action. Afterwards, we then study the normaliser of the Heisenberg group, which is its group of automorphisms. To finish, we apply these results to our previously determined models of the elliptic curves.

2 Elliptic Curves over $\mathbb{C}$

The aim of this section is to introduce elliptic curves over $\mathbb{C}$ and to make firm the link between them and complex tori, which serves as our motivation to study theta functions and the elliptic curves they give rise to in the subsequent sections. To avoid unnecessary obfuscation from the point we are trying to make here, we will quote most of the results here without proof, unless such a proof offers an enlightening discussion to our theory. That said, most of this section follows [Sil09] and [Har77].

2.1 The Theorem of Riemann-Roch

In this section, we shall use the word *curve* to mean a complete, non-singular curve over the field $\mathbb{C}$. For such a curve, we define the *divisor group of C*, denoted by $\mathrm{Div}(C)$, to be the free abelian group generated by points of C. So a divisor is a formal sum

$$D = \sum_{P \in C} n_P[P]$$

with $n_P \in \mathbb{Z}$, such that $n_P = 0$ for all but finitely many points $P \in \mathbb{C}$, and its *degree* is $\deg D := \sum n_P$. If we assume the curve C is smooth, and let $f \in \mathbb{C}(C)^*$, then we can associate to f the divisor $\mathrm{div}(f)$ given by

$$\mathrm{div}(f) := \sum_{P \in C} \mathrm{ord}_P(f)[P],$$

and we call any divisor $D \in \mathrm{Div}(C)$, such that $D = \mathrm{div}(f)$ for some $f \in \mathbb{C}(C)^*$, a *principal divisor*. The set of all principal divisors form a subgroup of $\mathrm{Div}(C)$, which we denote $\mathrm{Prin}(C)$. Any two divisors are *linearly equivalent*, written $D_1 \sim D_2$, if $D_1 - D_2$ is principal. The *Picard group* of C, denoted $\mathrm{Pic}(C)$, is defined as the quotient of $\mathrm{Div}(C)$ by the subgroup $\mathrm{Prin}(C)$, that is

$$\mathrm{Pic}(C) := \mathrm{Div}(C)/\,\mathrm{Prin}(C).$$

A divisor $D = \sum n_P[P]$ on C is *effective* if $n_P \geq 0$ for all $P \in C$, and for two divisors D_1, D_2, we denote $D_1 \geq D_2$ if their difference $D_1 - D_2$ is effective. The *space of (meromorphic) differential forms* on C is denoted by Ω_C, and to each $\omega \in \Omega_C$ we can associate to it the divisor

$$\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega)[P] \in \mathrm{Div}(C).$$

For any non-zero $\omega \in \Omega_C$, and divisor in the class of the image of $\mathrm{div}(\omega)$ in $\mathrm{Pic}(C)$ is called a *canonical divisor*, and is denoted K_C.

Definition 2.1 ([Sil09]). *The* Riemann-Roch *space of a divisor D on a curve C is the $\mathbb{C}$-vector space*

$$\mathcal{L}(D) := \{f \in \mathbb{C}(C)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

We have the following proposition:

Proposition 2.2 ([Har77]). *Let $D \in \mathrm{Div}(C)$.*

1. *If $\deg D < 0$, then*

$$\mathcal{L}(D) = \{0\} \qquad \textit{and} \qquad l(D) = 0.$$

2. *$\mathcal{L}(D)$ is a finite dimensional $\mathbb{C}$-vector space.*

3. *If $D' \in \mathrm{Div}(C)$ is linearly equivalent to D, then*

$$\mathcal{L}(D) \cong \mathcal{L}(D'), \qquad \text{and so} \qquad l(D) = l(D').$$

Theorem 2.3 (Riemann-Roch, [Har77])**.** *Let C be a smooth curve and let K_C be a canonical divisor of C. There is an integer $g \geq 0$, called the* genus of C, *such that for every $D \in \mathrm{Div}(C)$,*

$$l(D) - l(K_C - D) = \deg D - g + 1.$$

Corollary 2.4 ([Har77])**.** 1. *$l(K_C) = g$.*

2. *$\deg K_C = 2g - 2$.*

3. *If $\deg D > 2g - 2$, then $l(D) = \deg D - g + 1$.*

2.2 Elliptic Curves as Complex Tori

We now will apply these results to elliptic curves, namely:

Definition 2.5 ([Sil09])**.** *An elliptic curve E is a smooth curve of genus one, with a marked point $O \in E$.*

As a consequence of the Riemann-Roch Theorem 2.3, any elliptic curve E can be written as a Weierstrass equation in Legendre form:

Theorem 2.6 ([Har77])**.** *Any elliptic curve E can be written in Legendre form, that is*

$$E \equiv E_\lambda : y^2 = x(x-1)(x-\lambda) \tag{1}$$

for some $\lambda \in \mathbb{A}^1 \setminus \{0, 1\}$.

Proof. By assumption, E has a marked point O to which we can associate the divisor $D = [O]$. Then by the Riemann-Roch Theorem 2.3 and Proposition 2.2.1, we see that $l(nD) = n$ for all $n \geq 0$, since the genus of E is 1 and $\deg K_E = 0$ by Corollary 2.4.3. Now $l(0) = 1$ and $\mathcal{L}(0)$ consists of holomorphic functions without any poles, but the only holomorphic functions on E are necessarily constant, so $\mathcal{L}(0) = \mathbb{C}$. When $n \geq 1$, we have certain special cases:

$n = 1$: We have $l(D) = 1$. But $\mathcal{L}(D)$ definitely contains the constant functions which have no poles, so $\mathcal{L}(0) \subseteq \mathcal{L}(D)$. Moreover as $l(0) = l(D)$, this shows that E has no functions with just a simple pole, and that $\mathcal{L}(D) \cong \mathbb{C}$.

$n = 2$: Now $l(2D) = 2$, so we take $\{1, x\}$ to be a basis for $\mathcal{L}(2D)$, i.e. x has a double pole at O.

$n = 3$: Here $l(3D) = 3$, so we take $\{1, x, y\}$ to be basis for $\mathcal{L}(3D)$, where y has a triple pole at O.

$n = 4, 5$: When $l(4D) = 4$, we have that $\{1, x, y, x^2\}$ provides a basis for $\mathcal{L}(4D)$. Similarly when $l(5D) = 5$, $\{1, x, y, x^2, xy\}$ proves a basis for $\mathcal{L}(5D)$.

$n = 6$: Now the game changes; $l(6D) = 6$ but there are seven functions, $1, x, y, x^2, xy, x^3, y^2$ that belong to $\mathcal{L}(6D)$. It follows that there must be a linear relation between them:

$$a_1y^2 + a_2xy + a_3y = a_4x^3 + a_5x^2 + a_6x + a_7, \tag{2}$$

for $a_1, \ldots, a_7 \in \mathbb{C}$. Moreover, the coefficients a_1 and a_4 must be non-zero, since the functions y^2 and x^3 both have a six-fold pole at O and no other linear combination of the functions can provide that. As $\mathrm{char}(\mathbb{C}) = 0$, by completing the square in (2) it becomes an equation of the form $y^2 = f(x)$, where $f(x)$ is a cubic polynomial. Furthermore as $\mathbb{C}$ is algebraically closed, by considering the roots of $f(x)$ equation (2) can be transformed to a polynomial in the form

$$E_\lambda : y^2 = x(x - 1)(x - \lambda),$$

for some $\lambda \in \mathbb{A}^1 \setminus \{0, 1\}$. □

The assumption that E is non-singular asserts that $\lambda \notin \{0, 1\}$, and therefore the polynomial $f(x) = 0$ has distinct roots. Therefore the partial derivatives of $y^2 - f(x)$ do not vanish anywhere on E. From the identity $y^2 = f(x)$, we have the identity

$$2ydy = f'(x)dx.$$

Now dx/y is a holomorphic 1-form away from the points where $y = 0$, and in punctured neighbourhoods of such points we can instead write

$$\frac{dx}{y} = 2\frac{dy}{f'(x)},$$

since $f'(x)$ does not vanish since $f(x)$ has only simple roots. It follows that dx/y extends to a holomorphic 1-form ω on $E \setminus \{y = 0\}$. In fact, ω extends to a holomorphic 1-form on the whole of E, [Sil09]. The natural map

$$E(\mathbb{C}) \longrightarrow \mathbb{P}^1, \qquad (x, y) \longmapsto x,$$

is a double cover, ramified precisely over the four points $0, 1, \lambda, \infty \in \mathbb{P}^1$. To investigate the nature of such a map, let us for now consider instead the map

$$E(\mathbb{C}) \to \mathbb{C}, \qquad P \longmapsto \int_O^P \omega,$$

where the integral is along some path connecting O to P. This map however if not well-defined, since it depends on the choice of path connecting O to P. If $P = (x, y) \in E(\mathbb{C})$, we can alternatively view the map as happening in $\mathbb{P}^1$: we are trying to compute the complex line integral

$$P = (x, y) \longmapsto \int_\infty^x \frac{dt}{\sqrt{t(t-1)(t-\lambda)}}.$$

This integral is path-dependent due to the presence of the square root in the denominator which is not single valued, so really we have two copies of $\mathbb{P}^1$ to consider. However what we can do is make branch cuts, say connecting ∞ to 0 and 1 to λ, on each copy of $\mathbb{P}^1$ and glue them together. This way, away from these branch cuts we can choose one branch value of the square root. This construction of course topologically identifies the resulting elliptic curve as a torus, by realising that $\mathbb{P}^1$ is topologically a 2-sphere.

Returning back to the map

$$E(\mathbb{C}) \to \mathbb{C}, \qquad P \longmapsto \int_O^P \omega,$$

the path-dependence from the multi-valuedness of the square root can now be explained from integrating across the branch cuts in $\mathbb{P}^1$. The gluing of the two branch cuts on each copy of $\mathbb{P}^1$ gives rise to two non-contractible loops on the torus; let us label them α and β. We then obtain two complex numbers, the *periods* of E, given by

$$\omega_1 = \int_\alpha \omega, \qquad \omega_2 = \int_\beta \omega.$$

Moreover the paths α and β generate the first homology group of the associated torus, or equivalently $H_1(E, \mathbb{Z})$. Hence any two paths from O to P differ by a path homologous to $n_1\alpha + n_2\beta$ for some integers $n_1, n_2 \in \mathbb{Z}$. Thus the integral $\int_O^P \omega$ is well-defined up to the addition of the norm $n_1\omega_1 + n_2\omega_2$, suggesting that we look at the set

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}.$$

After this discussion, we now have a well-defined map

$$F : E(\mathbb{C}) \longrightarrow \mathbb{C}/\Lambda, \qquad P \longmapsto \int_O^P \omega \pmod{\Lambda}.$$

Further, the Λ is clearly a subgroup of $\mathbb{C}$, so the quotient group $\mathbb{C}/\Lambda$ is a group. Moreover as ω is translation invariant, F can be verified to be a group homomorphism:

$$\int_O^{P+Q} \omega \equiv \int_O^P \omega + \int_P^{P+Q} \omega \equiv \int_O^P \omega + \int_O^Q \tau_P^* \omega \equiv \int_O^P \omega + \int_O^Q \omega \pmod{\Lambda}.$$

Definition 2.7 ([Sil09]). *A* lattice Λ *in the complex numbers* $\mathbb{C}$ *is a discrete subgroup of the form* $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$*, where* ω_1 *and* ω_2 *are linearly independent over* $\mathbb{R}$*. A* complex torus *is a quotient group* $\mathbb{C}/\Lambda$ *of the complex plane by a lattice, with the projection* $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$*.*

Hence we see that we have very nearly shown that $E(\mathbb{C})$ is admits a complex analytic group homomorphism to the complex torus $T = \mathbb{C}/\Lambda$, provided that the periods ω_1 and ω_2 are linearly independent over $\mathbb{R}$. It turns out that the periods arising from this construction are linearly independent over $\mathbb{R}$, but we require some more machinery before addressing this.

2.3 Complex Tori as Elliptic Curves

This section will be dedicated to the inverse problem to the previous section, that is, given a lattice $\Lambda \subset \mathbb{C}$, how can one construct an elliptic curve? The answer is via elliptic functions:

Definition 2.8. *[Sil09] An* elliptic function (relative to the lattice Λ) *is a meromorphic function* $f(z)$ *on* $\mathbb{C}$ *that satisfies*

$$f(z+\omega) = f(z) \qquad \text{for all } z \in \mathbb{C} \text{ and all } \omega \in \Lambda.$$

Denote by $\mathbb{C}(\Lambda)$ the set of all elliptic functions relative to Λ. It can be shown that $\mathbb{C}(\Lambda)$ is in fact a field, [Sil09].

Definition 2.9 ([Sil09]). *A* fundamental parallelogram Π *for* Λ *is a set of the form*

$$\Pi = \{a + t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\},$$

where $a \in \mathbb{C}$ *and* $\{\omega_1, \omega_2\}$ *is a basis for* Λ*.*

Here we prove the complex analytic analogue that a meromorphic function with no poles is constant:

Proposition 2.10 ([Sil09]). *A holomorphic elliptic function is necessarily constant. Similarly, an elliptic function with no zeros is constant.*

Proof. Suppose that $f(z) \in \mathbb{C}(\Lambda)$ is holomorphic, and let D be a fundamental parallelogram for Λ. The periodicity of f implies that

$$\sup_{z\in\mathbb{C}} |f(z)| = \sup_{z\in\bar{D}} |f(z)|.$$

The set $\bar{D}$ is compact, so $|f(z)|$ is bounded on $\bar{D}$. Therefore f is in fact a bounded entire function, so Liouville's theorem tells us that f is constant. This proves the first statement. Finally, if f has no zeros, then $1/f$ has no poles and the previous argument applies. □

To circumvent this problem, we must introduce meromorphic functions with poles.

Definition 2.11 ([Sil09]). *Let $\Lambda \subset \mathbb{C}$ be a lattice. The* Weierstrass $\wp$-function (relative to Λ) *is defined by the series*

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

The Eisenstein series of weight $2k$ (for Λ) *is the series*

$$G_{2k}(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-2k}.$$

We state the following theorem without proof:

Theorem 2.12 ([Sil09]). *Let $\Lambda \subset \mathbb{C}$ be a lattice.*

(a) The Eisenstein series $G_{2k}(\Lambda)$ is absolutely convergent for all $k > 1$.

(b) The series defining the Weierstrass $\wp$-function converges absolutely and uniformly on every compact subset of $\mathbb{C} \setminus \Lambda$. The series defines a meromorphic function on $\mathbb{C}$ having a double pole with residue 0 at each lattice point and no other poles.

(c) The Weierstrass $\wp$-function is an even elliptic function.

(d) For all $z \in \mathbb{C} \setminus \Lambda$, the Weierstrass $\wp$-function and its derivative satisfy the relation

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6. \tag{3}$$

With all this at hand, we can finally make the connection between elliptic curves over $\mathbb{C}$ and complex tori;

Proposition 2.13 ([Sil09]). *Let $E/\mathbb{C}$ be an elliptic curve with Weierstrass coordinate functions x and y.*

(a) *Let α and β be closed paths on $E(\mathbb{C})$ that form a basis for $H_1(E,\mathbb{Z})$. Then the periods*

$$\omega_1 = \int_\alpha \frac{dx}{y}, \quad \textit{and} \quad \omega_2 = \int_\beta \frac{dx}{y}$$

are $\mathbb{R}$-linear independent, and hence form a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$.

(b) *Let Λ be the lattice generated by ω_1 and ω_2. Then the map*

$$F : E(\mathbb{C}) \longrightarrow \mathbb{C}/\Lambda, \qquad F(P) = \int_O^P \frac{dx}{y} \ (\textit{mod } \Lambda),$$

is a complex analytic isomorphism of Lie groups.

Proof. (a) There exists some lattice Λ_1 such that the map

$$\phi_1 : \mathbb{C}/\Lambda_1 \longrightarrow E(\mathbb{C}), \qquad \phi_1(z) = [\wp(z;\Lambda_1) : \wp'(z;\lambda_1) : 1],$$

is a complex analytic isomorphism. It follows that $\phi_1^{-1} \circ \alpha$ and $\phi_1^{-1} \circ \beta$ are a basis for $H_1(\mathbb{C}/\Lambda_1,\mathbb{Z})$, where we view α and β as maps $\alpha, \beta : S^1 \to E(\mathbb{C})$. We observe that $H_1(\mathbb{C}/\Lambda,\mathbb{Z})$ is natural isomorphic to the lattice Λ_1 via the map $\gamma \mapsto \int_\lambda dz$, while the differential dx/y on E pulls back to

$$\phi_1^*\left(\frac{dx}{y}\right) = \frac{d\wp(z)}{\wp'(z)} = dz \qquad \text{on } \mathbb{C}/\Lambda_1.$$

Therefore the periods

$$\omega_1 = \int_\alpha \frac{dx}{y} = \int_{\phi_1^{-1}\circ\alpha} dz \quad \text{and} \quad \omega_2 = \int_\beta \frac{dx}{y} = \int_{\phi_1^{-1}\circ\beta} dz$$

are a basis for Λ_1, so in particular they are linearly independent.

(b) We have just shown that the lattice Λ_1 corresponding to E is precisely the lattice generated by the periods of E. The composition $F \circ \phi$ then gives an analytic map

$$F \circ \phi : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda, \qquad (F \circ \phi)(z) = \int_O^{(\wp(z),\wp'(z))} \frac{dx}{y}.$$

Since

$$F^*(dz) = \frac{dx}{y} \quad \text{and} \quad \phi^*\left(\frac{dx}{y}\right) = \frac{d\wp(z)}{\wp'(z)} = dz,$$

we see that

$$(F \circ \phi)^* dz = dz.$$

On the other hand, any analytic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is of the form $\psi_a(z) = az$ for some number $a \in \mathbb{C}^*$. Since $\psi_a^*(z) = adz$, we see that $(F \circ \phi)(z) = z$, that is, the composition $F \circ \phi$ is just the identity map. But we know that ϕ is an analytic isomorphism, and consequently $F = \phi^{-1}$ is too.

□

With the correspondence between elliptic curves and complex tori firmly established, we can easily deduce the following:

Proposition 2.14. *[Sil09] Let $E/\mathbb{C}$ be an elliptic curve and let $n \geq 1$ be an integer.*

(a) There is an isomorphism of abstract groups

$$E[n] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

(b) The multiplication-by-n map $[n] : E \to E$ has degree n^2.

Proof. (a) Since $E(\mathbb{C})$ is isomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$, we have

$$E[n] \cong \left(\frac{\mathbb{C}}{\Lambda}\right)[n] \cong \frac{\frac{1}{n}\Lambda}{\Lambda} \cong \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^2.$$

(b) As $\mathrm{char}(\mathbb{C}) = 0$ and the map $[n]$ is unramified, the degree of $[n]$ is equal to the number of points in $E[n] = [n]^{-1}\{O\}$.

□

3 Theta Functions

We now want to find a non-constant map $\phi : E = \mathbb{C}/\Lambda \to \mathbb{P}^n$ to embed our elliptic curve E in projective space for some n. Recall that the complex projective space $\mathbb{P}^n$ is defined as the set of non-zero vectors in $\mathbb{C}^{n+1}$ up to multiplication of some non-zero scalar. As a complex manifold, $\mathbb{P}^n$ is covered by $n+1$ subsets $U_i = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_i \neq 0\}$, and a holomorphic map $\phi : E \to \mathbb{P}^n$ is defined, after composition with the natural map $\mathbb{C} \to \mathbb{C}/\Lambda$, by $n+1$ holomorphic functions $f_0, \ldots f_n$ on $\mathbb{C}$. These $n+1$ functions need not be periodic with respect to Λ, but must satisfy the weaker property, namely that they must be *theta functions*:

Definition 3.1 ([Dol97]). *A holomorphic function $f(z)$ on $\mathbb{C}$ is called a* theta function (relative to the lattice Λ) *if, for any $\lambda \in \Lambda$ there exists an invertible holomorphic function $e_\lambda(z)$ such that*

$$f(z+\lambda) = e_\lambda(z) f(z) \qquad \text{for all } \lambda \in \Lambda.$$

The set $\{e_\lambda(z)\}_{\lambda\in\Lambda}$ is called the theta factor *for f.*

As our first example of a theta function, we consider the *Riemann theta function* $\vartheta(z,\tau)$ on the lattice Λ_τ, defined by

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i(2zn + n^2\tau)\right).$$

It can easily be shown to converge uniformly on $\mathbb{C}\times\mathcal{H}$, see e.g. [Mum83], and it satisfies

$$\vartheta(z+m+n\tau,\tau) = \exp(-\pi i(2nz+n^2\tau))\vartheta(z,\tau),$$

so $\vartheta(z,\tau)$ is a theta function with the theta factor

$$e_{m+n\tau}(z) = \exp(-\pi i(2nz+n^2\tau)).$$

Now the question is what sort of form does a theta function $f(z)$ take on? First of all, let us consider the lattice $\Lambda_\tau = \{n+m\tau : n,m\in\mathbb{Z},\ \operatorname{Im}\tau>0\}$; any lattice Λ is equivalent to such a Λ_τ by homothety. Secondly, Liouville's theorem tells us that $f(z)$ cannot be doubly-periodic with respect to Λ_τ, so now we focus our attention to finding entire functions $f(z)$ with the simplest quasi-periodic behaviour with respect to Λ_τ, namely that $f(z)$ behaves as

$$f(z+1) = f(z), \qquad f(z+\tau) = \mathbf{e}(-(az+b))\cdot f(z),$$

where we write $\mathbf{e}(z) = \exp(2\pi iz)$, and which we call the functional equations for $f(z)$. Since $f(z)$ is periodic in z with respect to $z\mapsto z+1$, we can expand it as a Fourier series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n\mathbf{e}(nz), \qquad a_n\in\mathbb{C}.$$

Then writing $f(z+1+\tau)$ in terms of $f(z)$ by combining the functional equations in either order, we find that

$$f(z+1+\tau) = f(z+\tau) = \mathbf{e}(-(az+b)),$$

and also

$$f(z+1+\tau) = \mathbf{e}(-(a(z+1)+b))f(z+1) = \mathbf{e}(-a)\mathbf{e}(-(az+b))f(z),$$

so $a=k$ for some $k\in\mathbb{Z}$. Substituting the Fourier series into the second functional equation, we find that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} a_n\mathbf{e}(n\tau)\cdot\mathbf{e}(nz) &= f(z+\tau)\\ &= \mathbf{e}(-(kz+b))\cdot f(z)\\ &= \sum_{n\in\mathbb{Z}} a_n\mathbf{e}((n-k)z)\cdot\mathbf{e}(-b)\\ &= \sum_{n\in\mathbb{Z}} a_{n+k}\mathbf{e}(-b)\cdot\mathbf{e}(nz).\end{aligned}$$

Comparing the coefficients of the first and last term, we get the recursive relation

$$a_{n+k} = a_n\mathbf{e}((n\tau+b)). \tag{4}$$

Now if $k=0$, then for at most one n we have that $a_n \neq 0$, and we have the uninteresting possibility that $f(z)=\mathbf{e}(z)$. If $k\neq 0$, then there is the recursive relation for solving for a_{n+pk} in terms of a_n for all $p\in\mathbb{Z}$, but when $k\leq -1$ we see that the recursive relation leads to rapidly growing coefficients a_n, and so there cannot be any entire functions $f(z)$. However, when $k\geq 1$ this is not the case, and we find a k-dimensional vector space of possibilities for $f(z)$, as each $f(z)$ is determine by its Fourier coefficients $a_0,\ldots,a_{k-1}$. In fact, we can solve the recursive equation (4) explicitly; to simplify things, let us replace $f(z)$ by $f(z+\tau/2-b/k)$, then

$$\begin{aligned}f(z+\tau/2-b/k+\tau) &= \mathbf{e}(-(k(z+\tau/2-b/k)+b))\cdot f(z+\tau/2-b/k)\\ &\mathbf{e}(-k(z+\tau/2))\cdot f(z+\tau/2-b/k),\end{aligned}$$

so we may assume that $b=k\tau/2$. Then in letting $i\in\{0,\ldots,k-1\}$, we get [Dol97]

$$a_{i+pn} = \mathbf{e}\big(\tfrac{1}{2}\big((i+pn)^2\tau\big)/n\big)\cdot a_i$$

as our explicit solution to the recurrence relation (4). This shows that each $f(z)$ with the theta factor

$$e_{k\tau+l}(z) = \mathbf{e}\big(-n(kz+\tfrac{k^2}{2}\tau)\big) \tag{5}$$

can be written in the form

$$f(z) = \sum_{i=0}^{n-1} c_i\cdot\Theta_i(z,\tau)_n,$$

where

$$\Theta_i(z,\tau)_n = \sum_{r\in\mathbb{Z}} \mathbf{e}\big(\tfrac{1}{2}(i+rn)^2\tau/n\big)\cdot\mathbf{e}(z(i+rn)), \qquad i=0,\ldots,n-1.$$

We will see shortly that it will be more convenient to rewrite these functions in the form

$$\Theta_i(z,\tau)_n = \sum_{r\in\mathbb{Z}} \mathbf{e}\Big(\tfrac{1}{2}\big(\tfrac{i}{n}+r\big)^2 n\tau\Big)\cdot \mathbf{e}\Big(nz\big(\tfrac{i}{n}+r\big)\Big). \tag{6}$$

It is easy to see using the uniqueness of the Fourier coefficients for a holomorphic function that the $\Theta_i(z)_n$ are linearly independent, and hence form a basis for the theta functions with the theta factor (5). In summary,

Proposition 3.2 ([Dol97])**.** *Each theta factor is equivalent to the theta factor of the form*

$$e_{k+l\tau}(z) = \mathbf{e}\Big(-n\big(lz+\tfrac{l^2}{2}\tau\big)\Big).$$

The $\mathbb{C}$-vector space $R_n(\Lambda_\tau)$ of such functions is zero dimensional if $n<0$. For $n=0$ it consists of constant functions, whereas for $n>0$ is if of dimension n as is spanned by the functions

$$\Theta_i(z,\tau)_n = \sum_{r\in\mathbb{Z}} \mathbf{e}\Big(\tfrac{1}{2}\big(\tfrac{i}{n}+r\big)^2 n\tau\Big)\cdot \mathbf{e}\Big(nz\big(\tfrac{i}{n}+r\big)\Big).$$

A very useful definition is the following generalisation of the $\vartheta(z,\tau)$:

Definition 3.3 ([Mum83])**.** *For every $(a,b)\in\mathbb{Q}^2$ and $(z,\tau)\in\mathbb{C}\times\mathcal{H}$, the* theta function of rational characteristic *(a,b), is the series*

$$\vartheta\begin{bmatrix} a\\ b\end{bmatrix}(z,\tau) = \sum_{r\in\mathbb{Z}} \mathbf{e}\Big((r+a)(z+b)+\frac{1}{2}(r+a)(r+b)\tau\Big),$$

which will be commonly abbreviated at $\vartheta_{a,b}(z,\tau)$.

These are really just translates of $\vartheta(z,\tau)$ multiplied by an elementary exponential factor:

$$\vartheta\begin{bmatrix} a\\ b\end{bmatrix}(z,\tau) = \mathbf{e}\big(a(z+b)+\tfrac{1}{2}a^2\tau\big)\cdot\vartheta(z+a\tau+b), \qquad \text{for all } a,b\in\mathbb{Q},$$

and in terms of a theta function with rational characteristics, we now see that

$$\Theta_i(z,\tau)_n = \vartheta\begin{bmatrix} \frac{i}{n}\\ 0\end{bmatrix}(nz,n\tau), \tag{7}$$

and we call such functions *theta functions of weight n*, [Mum83]. With this in hand, we can rephrase Proposition 3.2 in the following terms:

Proposition 3.4 ([Mum83])**.** *Fix a lattice $\Lambda_\tau\subset\mathbb{C}$. Then a basis of the vector space of* *theta functions of weight n, $R_n(\Lambda_\tau)$, can be given by:*

$$x_i(z) = \vartheta \begin{bmatrix} \frac{i}{n} \\ 0 \end{bmatrix} (nz, n\tau), \qquad \textit{for } i \in \mathbb{Z}/n\mathbb{Z}.$$

Now we can determine the zeros of the $\vartheta \begin{bmatrix} \frac{i}{n} \\ 0 \end{bmatrix} (nz, n\tau)$:

Proposition 3.5 ([Dol97]). *A non-zero function $f(z) \in R_n(\Lambda_\tau)$ has exactly n zeros in $\mathbb{C}/\Lambda_\tau$ counting multiplicities.*

Proof. It is well known that the number of zeros (with multiplicity) of a holomorphic function $f(z)$ on an open subset U of $\mathbb{C}$ inside of a compact set $K \subset U$ is equal to

$$\#\text{ of zeros of } f(z) = \frac{1}{2\pi i} \int_{\partial K} d \log f(z)\ dz. \tag{8}$$

We assume that $f(z)$ has no zeros on ∂K, and after a suitable translation by $z_0 \in \mathbb{C}$, we can take the fundamental parallelogram $z_0 + \Pi$ for Λ_τ as our K. As $f(z) \in R_n(\Lambda_\tau)$,

$$d \log f(z + \tau) = -2\pi i n dz + d \log f(z)$$

as τ is fixed, and then we obtain from the above equation that

$$\begin{aligned} 2\pi i \cdot (\#\text{ of zeros of } f(z)) &= \int_{\partial K} d \log f(z)\ dz \\ &= \int_{z_0}^{z_0+1} (d \log f(z) - d \log f(z+\tau))\ dz \\ &- \int_{z_0}^{z_0+\tau} (d \log f(z) - d \log f(z+1))\ dz \\ &= \int_{z_0}^{z_0+1} 2\pi i n\ dz = 2\pi i n, \end{aligned}$$

which proves our assertion. □

Lemma 3.6 ([Dol97]). *The zero of the function $\vartheta_{a,b}(z, \tau)$ in $\mathbb{C}/\Lambda_\tau$ is the point*

$$P = \left(a + \frac{1}{2}\right)\tau + \left(b + \frac{1}{2}\right). \tag{9}$$

Proof. We observe that

$$\begin{aligned}
\vartheta_{\frac{1}{2},\frac{1}{2}}(-z,\tau) &= \sum_{m\in\mathbb{Z}} \mathbf{e}\big(\tfrac{1}{2}(m+1/2)^2\tau + (m+1/2)(-z+1/2)\big) \\
&= \sum_{k\in\mathbb{Z}} \mathbf{e}\big(\tfrac{1}{2}(-k-1/2)^2\tau + (k+1/2)(z-1/2)\big) \qquad \text{(where } k=-m-1\text{)} \\
&= \sum_{k\in\mathbb{Z}} \mathbf{e}\big(\tfrac{1}{2}(k+1/2)^2\tau + (k+1/2)(z+1/2)\big)\mathbf{e}(-(k+1/2)) \\
&= -\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau),
\end{aligned}$$

so the theta function $\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau)$ is an odd function, thus its zero is located at $z=0$ in Λ_τ. The zero point P for $\vartheta_{a,b}(z,\tau)$ is then the one stated, as $\vartheta_{a,b}(z,\tau)$ is obtained from $\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau)$ by translation. □

Corollary 3.7 ([Dol97])**.** *The zeros of the function $\vartheta_{a,b}(nz,n\tau)$ in $\mathbb{C}/\Lambda_\tau$ are the points*

$$P_i = \left(a+\frac{1}{2}\right)\tau + \frac{b}{n} + \frac{1}{2n} + \frac{i}{n}, \qquad i=0,\dots,n-1. \tag{10}$$

Proof. By Lemma 3.6, if P is the zero for $\vartheta_{a,b}(z,\tau)$ then a zero point for $\vartheta_{a,b}(nz,n\tau)$ is of the form

$$nP = \left(a+\frac{1}{2}\right)n\tau + \left(b+\frac{1}{2}\right) + \mathbb{Z} + n\tau\mathbb{Z},$$

and consequently

$$P_i = \left(a+\frac{1}{2}\right)\tau + \left(\frac{b}{n} + \frac{1}{2n} + \frac{i}{n}\right) + \Lambda_\tau, \qquad i=0,\dots,n-1. \tag{11}$$

□

We can now state and prove the main theorem of this section, that lets us embed a complex elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ into projective space by means of theta functions.

Theorem 3.8 ([Dol97, Mum83])**.** *For each $n \geq 1$, the map*

$$\begin{aligned}
\phi_n : E_\tau &\longrightarrow \mathbb{P}^{n-1} \\
z &\longmapsto \left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}(nz,n\tau), \vartheta\begin{bmatrix}\frac{1}{n}\\0\end{bmatrix}(nz,n\tau), \dots, \vartheta\begin{bmatrix}\frac{n-1}{n}\\0\end{bmatrix}(nz,n\tau)\right)
\end{aligned}$$

defines a holomorphic map. If $n \geq 3$, this map is a holomorphic embedding.

Proof. Firstly, the map is well-defined, since each theta function $\vartheta_{i/n,0}(nz,n\tau)$ has the same theta factor. Also from Corollary 3.7, they do not all vanish at the same point, hence define the same point in projective space. The map is holomorphic since the theta functions are holomorphic functions.

Let us show that it is injective when $n \geq 3$. Suppose that $\phi_n(z_1) = \phi_n(z_1')$, or that $d\phi_n(z_1) = 0$. Then for any integers k, l,

$$\vartheta \begin{bmatrix} \frac{i}{n} \\ 0 \end{bmatrix} (nz + k + l\tau, n\tau) = \mathbf{e}(ki/s)\mathbf{e}(2nlz + \tfrac{ln\tau}{2})\vartheta \begin{bmatrix} \frac{i}{n} \\ 0 \end{bmatrix} (nz, n\tau).$$

This shows that $\phi_n(z_1 + \frac{k}{n} + \frac{l}{n\tau}) = \phi_n(z_1' + \frac{k}{n} + \frac{l}{n}\tau)$. Note that, if $n \geq 3$, we can always choose k and l to be such that the four points z_1, z_1', $z_2 = z_1 + \frac{k}{n} + \frac{l}{n}\tau, z_2' = z_1 + \frac{k}{n} + \frac{l}{n}\tau$ are distinct. The linear space generated by the functions $\vartheta_{i/n,0}$ is of dimension n. So we can find a linear combination f of these functions such that it vanishes at z_1, z_2, and some other $n-3$ points $z_3, \ldots, z_{n-1}$, which are distinct modulo Λ_τ. But then f also vanishes at z_1' and z_2', or f has a double zero at z_1 and z_2. Thus we have $n+1$ zeros of f counting multiplicities, which contradicts Lemma 3.6 and proves our assertion. □

We finish this section by stating the image of our complex elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ is in fact and algebraic variety, when embedded with some projective space $\mathbb{P}^n$. This is due to Chow's Theorem, which states:

Theorem 3.9 (Theorem of Chow, [Mum74])**.** *Let X be a complete algebraic variety and Y a closed analytic subset of X_{hol}, where X_{hol} is the canonically associated analytic space structure on the underlying set of X. Then Y is Zariski closed in X.*

Chow proved the theorem for $X = \mathbb{P}^n$, and as far as we are concerned this is the case we are after, with $Y = \phi_n(E_\tau)$.

4 Projectively Embedded Elliptic Curves

We just saw that a basis for the space $R_n(\Lambda_\tau)$ of theta functions of weight n is given by

$$x_i(z,\tau) = \vartheta \begin{bmatrix} \frac{i}{n} \\ 0 \end{bmatrix} (nz, n\tau), \qquad \text{for } i \in \mathbb{Z}/n\mathbb{Z}, \tag{12}$$

and that when $n \geq 3$, the map

$$\phi_n : z \longmapsto [x_0(z) : \ldots, x_{n-1}(z)]$$

is a holomorphic embedding of the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ into $\mathbb{P}^{n-1}$. For any Λ_τ-quasi-periodic function $f(z)$, define the two transformations

$$\begin{aligned} (S_b f)(z) &= f(z+b), \\ (T_a f)(z) &= \mathbf{e}(iaz + a^2\tau/2) \cdot f(z + a\tau), \end{aligned} \tag{13}$$

for any real numbers a and b. Then in terms of these operations we have

$$\begin{aligned} \vartheta_{a,b}(z,\tau) &= (S_b T_a \vartheta_{0,0})(z,\tau), \\ (S_{b_1}\vartheta_{a,b})(z) &= \vartheta_{a,b+b_1}(z,\tau), \qquad \text{for } a, b_1, b \in \mathbb{Z}/n\mathbb{Z}, \\ (T_{a_1}\vartheta_{a,b})(z,\tau) &= \mathbf{e}(-a_1 b)\cdot \vartheta_{a_1+a,b}(z,\tau), \qquad \text{for } a, a_1, b \in \mathbb{Z}/n\mathbb{Z}, \\ \vartheta_{a+p,b+q}(z,\tau) &= \mathbf{e}(aq)\cdot\vartheta_{a,b}(z,\tau), \qquad \text{for all } p,q\in\mathbb{Z}, a,b\in\mathbb{Z}/n\mathbb{Z}. \end{aligned} \tag{14}$$

In particular, the image of the map $\phi_n(z)$ in $\mathbb{P}^{n-1}$ must be invariant under the action of S_b and T_a. The following lemma provides an identity for our basis theta functions $x_i(z)$.

Lemma 4.1 ([Kra70, Tra85])**.** *For any positive integer n, and for all positive integers $a, b \in \mathbb{Z}$, we have the following identity*

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)\cdot\vartheta\begin{bmatrix} a \\ b+\frac{1}{n} \end{bmatrix}(z,\tau)\cdot\ldots\cdot\vartheta\begin{bmatrix} a \\ b+\frac{n-1}{n} \end{bmatrix}(z,\tau) = c\cdot\vartheta\begin{bmatrix} a \\ nb+\frac{n-1}{2} \end{bmatrix}(nz,n\tau), \tag{15}$$

for any (z,τ) in $\mathbb{C}\times\mathcal{H}$, where c is independent of z, a, and b.

Proof. From Lemma 3.6 and its Corollary 3.7, the n simple zeros of both sides of equation (15) coincide. So their ratio is an entire and bounded function, and thus equal to a constant c by Liouville's theorem. To see that c is independent of both a and b, observe that the ratio remains the same under $z \mapsto z + 1/n$ and $z \mapsto z + 1/n\tau$. □

4.1 The Hesse Cubic

To develop the general case of embedding our elliptic curve E_τ into some $\mathbb{P}^{n-1}$ via the theta functions $\vartheta_{i/n,0}(nz, n\tau)$, we start with the easiest case when $n = 3$. This method was originally done by Klein in [Kle92], as well as by Hurwitz [Hur86] and Bianchi [Bia80], where they used products of σ-functions with prescribed zeros in the fundamental parallelogram. We follow a similar vein, of course here we adapt their methods by using theta functions instead. The closest reference to ours using ϑ-functions is [Tra85].

Let $E_\tau = \mathbb{C}/\Lambda_\tau$ be an elliptic curve, and let $z \in E_\tau$. Consider the ϑ-products[1]:

$$\begin{aligned} x_0(z) &= \vartheta_{\frac{1}{2},\frac{1}{6}}(z,\tau)\cdot\vartheta_{\frac{1}{2},\frac{1}{3}+\frac{1}{6}}(z,\tau)\cdot\vartheta_{\frac{1}{2},\frac{2}{3}+\frac{1}{6}}(z,\tau), \\ x_1(z) &= \vartheta_{\frac{1}{3}+\frac{1}{2},\frac{1}{6}}(z,\tau)\cdot\vartheta_{\frac{1}{3}+\frac{1}{2},\frac{1}{3}+\frac{1}{6}}(z,\tau)\cdot\vartheta_{\frac{1}{3}+\frac{1}{2},\frac{2}{3}+\frac{1}{6}}(z,\tau), \\ x_2(z) &= \vartheta_{\frac{2}{3}+\frac{1}{2},\frac{1}{6}}(z,\tau)\cdot\vartheta_{\frac{2}{3}+\frac{1}{2},\frac{1}{3}+\frac{1}{6}}(z,\tau)\cdot\vartheta_{\frac{2}{3}+\frac{1}{2},\frac{2}{3}+\frac{1}{6}}(z,\tau). \end{aligned}$$

[1] The adjustment to the rational characteristics by the half-factor is so that the sum of their zeros add up to 0 mod Λ_τ, so that the x_i belong to the line bundle $\mathcal{L}(3[O])$ [Hul86].

By Lemma 4.1, we can rewrite each $x_i(z)$ as[2]

$$x_i(z) = \vartheta \begin{bmatrix} \frac{i}{3} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (3z, 3\tau),$$

so the $x_i(z)$ form a basis for the vector space $R_3(\Lambda_\tau)$. Moreover, by Lemma 3.6 and Corollary 3.7, the three zeros of each $x_i(z)$ are precisely

$$P_{ki} = \frac{1}{3}(k + i\tau), \qquad k, i = 0, 1, 2, \tag{16}$$

and their sum is $\sum_{k=0}^{2} P_{ki} \equiv 0 \bmod \Lambda_\tau$ for each $i = 0, 1, 2$, see Figure 1.

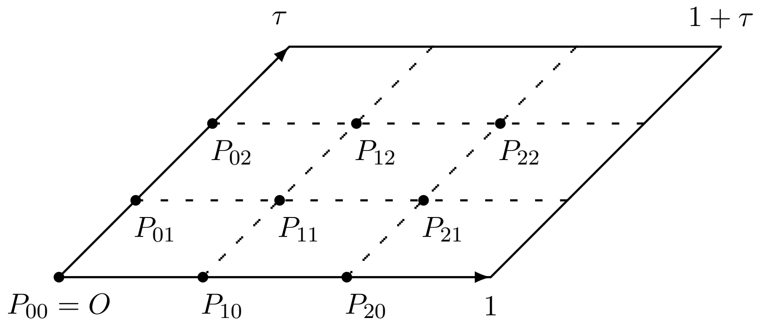

Figure 1: The zeros of $x_i(z)$ for $i = 0, 1, 2$ in the fundamental parallelogram.

Thus the three zeros belonging to each $x_i(z)$ lie on an inflection line. Denoting $\sigma = S_{\frac{1}{3}}$ and $\tau = T_{\frac{1}{3}}$, so that their action is then translating by a 3-torsion point of E_τ, we can easily verify that

$$\begin{aligned} \sigma(x_0) &\sim x_0, & \tau(x_0) &\sim x_1; \\ \sigma(x_1) &\sim \epsilon x_1, & \tau(x_1) &\sim x_2; \\ \sigma(x_2) &\sim \epsilon^2 x_2, & \tau(x_2) &\sim x_0, \end{aligned} \tag{17}$$

where $\epsilon = e^{2\pi i/3}$ and by the symbol $\sim$ we mean up to a common nowhere vanishing holomorphic function. Thus in $\mathbb{P}^2$, considering now the index $i \in \mathbb{Z}/3\mathbb{Z}$, the action of σ and τ becomes

$$\begin{aligned} \sigma &: x_i \longmapsto \epsilon^i x_i, \\ \tau &: x_i \longmapsto x_{i+1}. \end{aligned} \tag{18}$$

Similarly let $\imath : z \mapsto -z$ denote the involution on the elliptic curve E_τ, then since

[2]There is not actually much to gain from writing out the $x_i(z)$ as ϑ-products, since it is only after Lemma 4.1 that it becomes apparent they form a basis for $R_3(\Lambda_\tau)$. However as σ-products were first used in Klein and Fricke's treatise [Kle92], as well as in [Bia80, Hur86], starting with ϑ-products seems a fitting tribute to their century's old papers.

$\vartheta_{a,b}(-z,\tau) = \vartheta_{-a,-b}(z,\tau)$ its action extends to $\mathbb{P}^2$ via

$$\imath : x_i \longmapsto x_{-i}. \tag{19}$$

The cubic homogeneous polynomials that are invariant under S are x_0^3, x_1^3, x_2^3 and $x_0x_1x_2$, so then the image of E_τ in $\mathbb{P}^2$ has to be of the form [Tra85]

$$f(x_0,x_1,x_2) = Ax_0^3 + Bx_1^3 + Cx_3^3 + Dx_0x_1x_2 = 0.$$

Further, for f to be invariant under the action of T, we must have that $A = B = C$, so after rescaling and setting $D = -3a$ we arrive at

$$E_{a(\tau)}: \quad f(x_0,x_1,x_2) = x_0^3 + x_1^3 + x_2^3 - 3a(\tau)x_0x_1x_2 = 0, \tag{20}$$

which is the homogeneous cubic polynomial known as the *Hesse cubic*. The variable $a(\tau)$ depends on the lattice Λ_τ and it is well known that an elliptic E_τ in Hesse form is non-singular if and only if $a(\tau) \neq 1, \epsilon, \epsilon^2, \infty$, [AD06]. In the case that $a(\tau)$ does take on one of these four values, then the curve E_a degenerates into the union of three lines that form a triangle. We will investigate this scenario in the subsequent section.

Remark 4.2. *Consider the affine part of the Hesse pencil where $x_0(z) \neq 0$, then it is isomorphic to the curve C in $\mathbb{C}^2$ given by the equation*

$$C: \quad 1 + x_1^2 + x_2^2 - 3ax_1x_2 = 0.$$

It follows that the functions

$$\Phi_1(z) = \frac{x_1(z)}{x_0(z)}, \qquad \Phi_2(z) = \frac{x_2(z)}{x_0(z)},$$

define a surjective holomorphic map $\mathbb{C}^2 \setminus Z \to C$, where Z is the zero set of $\vartheta_{1/2,0}(3z,3\tau) = x_0(z)$, and the fibres of the map are equal to the cosets $z + \mathbb{Z} + \tau\mathbb{Z}$. Moreover, the functions $\Phi_1(z)$ and $\Phi_2(z)$ are elliptic functions with respect to Λ_τ, hence we have succeeded in parametrising the Hesse pencil by doubly-periodic functions [Dol97]. This of course is an example of the Uniformisation Theorem of Klein and Poincaré, (a consequence of which being) that any genus one algebraic curve can be parametrised by elliptic functions.

4.2 The General Case for $n > 3$

We now aim to generalise the results of the previous section, by embedding the basis of the $R_n(\Lambda_\tau)$ into $\mathbb{P}^{n-1}$ for any integer $n > 3$. The content of this section (and indeed the previous one) is fundamentally classical; the general case for odd and even n has been covered in Klein and Fricke's treatise [Kle92], whereas Bianchi goes into incredible depth for $n = 3, 5$ in [Bia80]. Hurwitz considers the case when n is even in [Hur86]. A modern reference is [Hul86], but all of the previous references use σ-products as already stated. In [Tra85], Tracy provides a sketch of the case when n is odd, so our main contribution is the even n case; as such, the author is unaware of an English translate of the proof of Theorem 4.3.

Using the previous example on the Hesse pencil as our guiding analogy, we first remark that n homogeneous polynomials $x_0, \dots, x_{n-1}$ each should have a zero at n unique n-torsion points. To this end, from Corollary 3.7 we know that the theta functions $\vartheta \begin{bmatrix} \frac{i}{n} + a \\ b \end{bmatrix}$, where $(a, b) \in \mathbb{Q}^2$, have n zeros precisely at the points at

$$P_{ik} = \left(\frac{i}{n} + a + \frac{1}{2}\right)\tau + \left(\frac{b}{n} + \frac{1}{2n} + \frac{k}{n}\right), \qquad k = 0, \dots, n-1,$$

and their sum is

$$\begin{aligned} \sum_{k=0}^{n-1} P_{ik} &= \left(i + na + \frac{n}{2}\right)\tau + \left(b + \frac{1}{2} + \frac{(n-1)}{2}\right) \\ &\equiv \left(na + \frac{n}{2}\right)\tau + \left(b + \frac{n}{2}\right) \mod \Lambda_\tau. \end{aligned} \tag{21}$$

We require that $\sum_{k=0}^{n-1} P_{ik} \equiv 0 \mod \Lambda_\tau$, so in setting

$$\begin{cases} a = b = 0, & \text{when } n \text{ is even,} \\ a = b = \frac{1}{2}, & \text{when } n \text{ is odd,} \end{cases}$$

this is achieved. This amounts to considering the following bases for $R_n(\Lambda_\tau)$:

$$x_i(z) = \begin{cases} \vartheta \begin{bmatrix} \frac{i}{n} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (nz, n\tau) = \prod_{k=0}^{n-1} \vartheta \begin{bmatrix} \frac{i}{n} + \frac{1}{2} \\ \frac{k}{n} + \frac{1}{2n} \end{bmatrix} (z, \tau), & n \equiv 1 \pmod 2, \\ \vartheta \begin{bmatrix} \frac{i}{n} \\ 0 \end{bmatrix} (nz, n\tau) = \prod_{k=0}^{n-1} \vartheta \begin{bmatrix} \frac{i}{n} \\ \frac{k}{n} \end{bmatrix} (z, \tau), & n \equiv 0 \pmod 2. \end{cases} \tag{22}$$

With this at hand, we are now ready to state the main result of this section:

Theorem 4.3. *Let Λ_τ be a lattice and let $E_\tau = \mathbb{C}/\Lambda_\tau$ be its corresponding elliptic curve. Then for $n \geq 4$, the image $\phi_n(E_\tau) \subseteq \mathbb{P}^{n-1}$ is the set-theoretic intersection of $\frac{n(n-3)}{2}$ quadrics.*

Proof. It is a classical result of the Riemann quartic identity (see [Kra70, Tra85]) that

$$\begin{aligned} &\vartheta_{1,1}(u_1+u_2,\tau)\cdot\vartheta_{1,1}(u_1-u_2,\tau)\cdot\vartheta_{1,1}(u_3+u_4,\tau)\cdot\vartheta_{1,1}(u_3-u_4,\tau)\\ &+\vartheta_{1,1}(u_1+u_3,\tau)\cdot\vartheta_{1,1}(u_1-u_3,\tau)\cdot\vartheta_{1,1}(u_4+u_2,\tau)\cdot\vartheta_{1,1}(u_4-u_2,\tau)\\ &+\vartheta_{1,1}(u_1+u_4,\tau)\cdot\vartheta_{1,1}(u_1-u_4,\tau)\cdot\vartheta_{1,1}(u_2+u_3,\tau)\cdot\vartheta_{1,1}(u_2-u_3,\tau)=0, \end{aligned} \tag{23}$$

for any $u_i \in E_\tau$, and where we write $\vartheta_{1,1}(z,\tau)$ in the place of $\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau)$. Written out in full, the identity is

$$\begin{aligned} &\mathbf{e}(u_3)\cdot\vartheta(u_1+u_2+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\cdot\vartheta(u_1-u_2+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\ &\quad\cdot\vartheta(u_3+u_4+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\cdot\vartheta(u_3-u_4+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\ &+\mathbf{e}(u_4)\cdot\vartheta(u_1+u_3+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\cdot\vartheta(u_1-u_3+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\ &\quad\cdot\vartheta(u_4+u_3+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\cdot\vartheta(u_4-u_3+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\ &+\mathbf{e}(u_2)\cdot\vartheta(u_1+u_4+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\cdot\vartheta(u_1-u_4+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\ &\quad\cdot\vartheta(u_2+u_3+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\cdot\vartheta(u_2-u_3+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)=0, \end{aligned} \tag{24}$$

after removing common nowhere vanishing factors. Making the substitutions

$$u_i \longmapsto \frac{nz}{2}+\frac{\alpha_i\tau}{n}-\gamma\cdot\left(\frac{\tau}{2}+\frac{1}{2}\right),$$

where

$$\gamma = 0 \quad \text{or} \quad = \frac{1}{2},$$

depending on whether n is odd or even, respectively, and letting $\tau \mapsto n\tau$ we arrive at

$$\begin{aligned} &c_1(\tau)\cdot\vartheta\Big(nz+\tfrac{\alpha_1+\alpha_2}{n}\tau+(\tfrac{1}{2}-\gamma)(n\tau+1),n\tau\Big)\cdot\vartheta\Big(\tfrac{\alpha_1-\alpha_2}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\ &\vartheta\Big(nz+\tfrac{\alpha_3+\alpha_4}{n}\tau+(\tfrac{1}{2}-\gamma)(n\tau+1),n\tau\Big)\cdot\vartheta\Big(\tfrac{\alpha_3-\alpha_4}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\ &+c_2(\tau)\cdot\vartheta\Big(nz+\tfrac{\alpha_1+\alpha_3}{n}\tau+(\tfrac{1}{2}-\gamma)(n\tau+1),n\tau\Big)\cdot\vartheta\Big(\tfrac{\alpha_1-\alpha_3}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\ &\cdot\vartheta\Big(nz+\tfrac{\alpha_2+\alpha_4}{n}\tau+(\tfrac{1}{2}-\gamma)(n\tau+1),n\tau\Big)\cdot\vartheta\Big(\tfrac{\alpha_2-\alpha_4}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\ &+c_3(\tau)\cdot\vartheta\Big(nz+\tfrac{\alpha_1+\alpha_4}{n}\tau+(\tfrac{1}{2}-\gamma)(n\tau+1),n\tau\Big)\cdot\vartheta\Big(\tfrac{\alpha_1-\alpha_4}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\ &\vartheta\Big(nz+\tfrac{\alpha_2+\alpha_3}{n}\tau+(\tfrac{1}{2}-\gamma)(n\tau+1),n\tau\Big)\cdot\vartheta\Big(\tfrac{\alpha_2-\alpha_3}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)=0, \end{aligned} \tag{25}$$

where c_1, c_2 and c_3 are nowhere vanishing functions independent of z, but depend upon $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and τ. Comparing the identity (25) to our basis (22), we find that

$$\begin{aligned} &c_1(\tau)\cdot y_{\alpha_1-\alpha_2}(\tau)\cdot y_{\alpha_3-\alpha_4}(\tau)\cdot x_{\alpha_1+\alpha_2}(z)\cdot x_{\alpha_3+\alpha_4}(z)\\ &+c_2(\tau)\cdot y_{\alpha_1-\alpha_3}(\tau)\cdot y_{\alpha_2-\alpha_4}(\tau)\cdot x_{\alpha_1+\alpha_3}(z)\cdot x_{\alpha_2+\alpha_4}(z)\\ &+c_3(\tau)\cdot y_{\alpha_1-\alpha_4}(\tau)\cdot y_{\alpha_2-\alpha_3}(\tau)\cdot x_{\alpha_1+\alpha_4}(z)\cdot x_{\alpha_2+\alpha_3}(z)=0, \end{aligned} \tag{26}$$

is a quadratic relation satisfied by the $x_i(z)$, where we have set

$$y_{\alpha_i-\alpha_j}(\tau) := \vartheta\Big(\tfrac{\alpha_i-\alpha_j}{n}\tau + \tfrac{1}{2}n\tau + \tfrac{1}{2}, n\tau\Big). \tag{27}$$

We now claim that equation (26) defines $\frac{n(n-3)}{2}$ linearly independent quadrics in total. To see this, notice that each quadratic term in equation (26), $x_\alpha x_\beta$ has the same index sum, $s = \alpha + \beta$. We have to consider now the cases when n is odd or even.

When n is odd, the quadrics whose sum equals a fixed s can be written out as

$$x_0x_s,\ x_1x_{s-1}, \dots,\ x_jx_{s-j}, \dots,\ x_nx_{s-n}, \tag{28}$$

and in total there are $\frac{n+1}{2}$ quadrics. However by the identity (26), any quadric whose index sum equals s can be written out as a linear combination of the first two quadrics of (28), that is

$$x_jx_{s-j} = a_jx_0x_s + b_jx_1x_{s-1}, \tag{29}$$

so there are in total $\frac{n-3}{2}$ linearly independent quadrics for each s when n is an odd integer, and as s ranges from 0 to $n-1$, we have $\frac{n(n-3)}{2}$ linearly independent quadrics in total.

When n is even, we have to distinguish between the cases when s is odd or even. In the former, there are $\frac{n}{2}$ quadrics with a fixed odd s:

$$x_0x_s,\ x_2x_{s-2},\ \dots\ ,\ x_{n-2}x_{s-n+2},$$

and again by (29) there are $\frac{n-4}{2}$ linearly independent quadric for a fixed odd s and a total of of $\frac{n}{2} \cdot \frac{n-4}{2} = \frac{n(n-4)}{4}$ independent quadrics.

When s is even, we consider again (29) for when j is odd, but instead when j is even we consider the equation

$$x_jx_{s-j} = c_jx_0x_s + d_jx_2x_{j-2} \tag{30}$$

that arises from (26). Now there are a total of $\frac{n}{2}+1$ quadrics for a fixed even s, and and the relations (29) and (30) mean that there are $\frac{n-2}{2}$ independent quadrics, thus a total of $\frac{n}{2} \cdot \frac{n-2}{2} = \frac{n(n-2)}{4}$ as s ranges from 0 to $n-2$. Hence for n even, there are

$$\frac{n(n-4)}{4} + \frac{n(n-2)}{4} = \frac{n(n-3)}{2}$$

linearly independent quadrics. □

Clearly, Theorem 4.3 does not tell us anything about the case when $n = 3$, since the subscripts in (26) have to be distinct to be non-trivial. The equations (26) were first found by Klein in [Kle92], and were also obtained independently by Vélu [Vél78] in his thesis when n is prime. In fact, Vélu found the set of quadrics in (26) under the assumption that the underlying field was of any characteristic not equal to n. Also if we set $n = 2d + 1$ for some $d \geq 2$, that is if $n \geq 5$ is odd, then Gross proved in [GP96] that the quadrics in (26) can be written as the rank 2 locus of the $n \times n$ matrix

$$M_d = \Big(y_{(d+1)(i-j)} \cdot x_{(d+1)(i+j)} \Big)_{i,j \in \mathbb{Z}/n\mathbb{Z}} \tag{31}$$

which we remark is an $n \times n$ skew-symmetric matrix as $y_0(\tau) = \vartheta_{1,1}(0, n\tau)$.

4.3 The Fermat Quartic

When $n = 4$, where are $\frac{4\cdot(4-3)}{2} = 2$ independent quadrics, which are

$$\begin{aligned} Q_0(x_0, \ldots, x_3) &= ax_0^2 + bx_2^2 + x_1x_3 = 0, \\ Q_1(x_0, \ldots, x_3) &= cx_1^2 + dx_3^2 + x_0x_2 = 0, \end{aligned} \tag{32}$$

for some parameters a, b, c and d whose sole dependence is on τ. In fact, applying our transformations S and T to the quadrics in (32), we find that $a = b = c = d$, so let us introduce a new variable $\lambda = 1/2a$ to find

$$\begin{aligned} Q_0(x_0, \ldots, x_3) &= x_0^2 + x_2^2 + 2\lambda x_1x_3 = 0, \\ Q_1(x_0, \ldots, x_3) &= x_1^2 + x_3^2 + 2\lambda x_0x_2 = 0. \end{aligned} \tag{33}$$

Now $x_0(z)$ vanishes at $z_0 = \frac{1}{2}(\tau + 1)$, so from Q_0 we find that

$$\lambda = -\frac{x_2^2(z_0)}{2x_1(z_0)x_3(z_0)} \qquad \text{with} \qquad z_0 = \frac{1}{2}(\tau + 1),$$

which agrees with Hulek's result in [Hul86], and is also one of the main examples given by Mumford in [Mum66]. Let us set

$$E_\lambda := Q_0(\lambda) \cap Q_1(\lambda), \tag{34}$$

then one has the following Proposition, whose proof is straightforward:

Proposition 4.4 ([BHM84]). *For all values $\lambda \in \mathbb{P}^1 \setminus \{0, \infty, \pm 1, \pm i\}$ the curve E_λ is a smooth elliptic curve. Otherwise E_λ is a connected cycle of four lines.*

A $\lambda \in \mathbb{P}^1$ varies, the curves E_λ sweep out a surface F. Eliminating the parameter λ from the quadratic equations one finds that the equation for F is

$$(x_1^2 + x_3^2)x_1x_3 - (x_0^2 + x_2^2)x_0x_2 = 0.$$

After a change of coordinates

$$x_0 \mapsto (ix_0 + x_2), \quad x_1 \mapsto (x_1 + ix_3),$$
$$x_2 \mapsto (ix_0 - x_2), \quad x_3 \mapsto (x_1 - ix_3),$$

the above equation is transformed (up to a constant) to the Fermat quartic

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,$$

that is that F is projectively equivalent to the Fermat quartic, [BHM84].

4.4 The Bianchi Quintic

Let us consider the case $n = 5$ in Theorem 4.3. There are then $\frac{5\cdot(5-3)}{2} = 5$ linearly independent quadrics:

$$\begin{aligned}
Q_0(x_0,\ldots,x_4) &= x_0^2 + ax_2x_3 + bx_1x_4,\\
Q_1(x_0,\ldots,x_4) &= x_1^2 + ax_3x_4 + bx_2x_0,\\
Q_2(x_0,\ldots,x_4) &= x_2^2 + ax_4x_0 + bx_3x_1,\\
Q_3(x_0,\ldots,x_4) &= x_3^2 + ax_0x_1 + bx_4x_2,\\
Q_4(x_0,\ldots,x_4) &= x_4^2 + ax_1x_2 + bx_0x_3,
\end{aligned} \tag{35}$$

where a and b are parameters that depend solely on τ. In fact, we can determine the relationship between a and b; as $x_0(z)$ vanishes at $z = 0$, evaluating the two quadrics Q_2 and Q_4 there yields

$$a = -\frac{x_4^2(0)}{x_1(0)x_2(0)}, \quad \text{and} \quad b = -\frac{x_2^2(0)}{x_3(0)x_1(0)}.$$

Then, since $x_1(z) = x_4(-z) = (\tau x_0)(-z) = -(\tau x_0)(z) = -x_4(z)$, and similarly $x_2(z) = -x_3(z)$, we see that $x_1(0) = -x_4(0)$ and $x_2(z) = -x_3(z)$, thus

$$a = -\frac{x_1(0)}{x_2(0)}, \qquad b = \frac{x_2(0)}{x_1(0)} = -\frac{1}{a}, \tag{36}$$

so the five quadrics (35) become

$$Q_i(x_0,\ldots,x_4) = x_i^2 + ax_{i+2}x_{i+3} - \frac{1}{a}x_{i+1}x_{i+4} = 0. \tag{37}$$

In fact, we can arrange the quadrics in (37) neatly in the form of a skew-symmetric 5×5 matrix

$$M = \begin{bmatrix} 0 & -x_3 & -ax_1 & ax_4 & x_2 \\ x_3 & 0 & -x_4 & -ax_2 & ax_0 \\ ax_1 & x_4 & 0 & -x_0 & -ax_3 \\ -ax_4 & x_2 & x_0 & 0 & -x_1 \\ -x_2 & -ax_0 & ax_3 & x_1 & 0 \end{bmatrix},$$

whose five 4×4 Pfaffians make up the quadrics (37). This arrangement of quadrics was first studied in great detail by Bianchi [Bia80], hence our naming of this example as the "Bianchi quintic".

5 The Heisenberg Group and Abstract Configurations

5.1 The Heisenberg Group and Schrödinger Representation

We largely follow [Hul86] for most of this section and we set $V = \mathbb{C}^n$ for brevity. We saw in the previous section that the action of the n-torsion subgroup $E_\tau[n]$ of the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ induced an action on $\mathbb{P}^{n-1} = \mathbb{P}(V)$. Consider the transformations[3]

$$\sigma : x_i \longmapsto x_{i-1},$$
$$\tau : x_i \longmapsto \epsilon^i x_i,$$

where $\epsilon = e^{2\pi i/n}$ is a primitive n-th root of unity and $i \in \mathbb{Z}/n\mathbb{Z}$. As an abstract group, $E_\tau[n] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and in particular is commutative. However the transformations σ and τ fail to commute, but only very little: $[\sigma, \tau] = \epsilon \cdot \mathrm{Id}_n$.

Definition 5.1 ([Hul86]). *The subgroup $H_n \subset \mathrm{GL}(V)$ generated by σ and τ is called the* Heisenberg group of level n. *The representation of H_n defined by the inclusion is called the* Schrödinger representation of the Heisenberg group.

Remark 5.2. *1. The centre of the Heisenberg group H_n equals*

$$\mu_n = \{\epsilon^m \cdot \mathrm{Id}_V : m \in \mathbb{Z}/n\mathbb{Z}\}$$

and the group H_n is a central extension

$$1 \longrightarrow \mu \longrightarrow H_n \longrightarrow \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

where σ and τ are mapped to $(1, 0)$ and $(0, 1)$ respectively. The order of H_n is n^3 and in fact, if $n = p \geq 3$ is a prime number then H_p is the unique group of order p^3

[3] For the purposes of this report and for the content of this section to be closer to the literature, we consider $\sigma(x_i) = x_{i-1}$ rather than $\sigma(x_i) = x_{i+1}$ as it was in the previous sections.

with exponent p.

2. *When n is odd, the Heisenberg group H_n actually lies within* $\mathrm{SL}(V)$.

3. *The commutator map induces a non-degenerate skew-symmetric bilinear form*

$$(\, , \,) : (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

given by

$$\epsilon^{(\alpha,\beta)} = \frac{1}{n} \operatorname{Tr}(\alpha\beta\alpha^{-1}\beta^{-1}). \tag{38}$$

4. *The Schrödinger representation of H_n is an irreducible representation of H_n, and if p is a prime number then we can describe all irreducible representations of H_p. To do this let*

$$\rho : H_p \to \mathrm{GL}(V)$$

be the Schrödinger representation, and denote the corresponding H_p-module by V^1. The Schrödinger representation gives rise to $p-1$ irreducible H_p-modules V^i, $i = 1, \ldots, p-1$ of dimension p in the following way:

$$\begin{aligned} \rho^i : H_p &\longrightarrow \mathrm{GL}(V), \\ \rho^i(\sigma) &:= \sigma, \\ \rho^i(\tau) &:= \tau^i. \end{aligned}$$

In addition, $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and hence also H_p has p^2 characters which we will denote by V^{kl} with $k, l \in \mathbb{Z}/p\mathbb{Z}$. Since the sum over the squares of the dimensions of the irreducible representations described is equal to

$$(p-1)p^2 + p^2 = p^3 = |H_p|,$$

this is a complete list of irreducible H_p-modules.

5. *As the Schrödinger representation is irreducible, it follows that our elliptic curves are in fact normal, i.e. they span their respective $\mathbb{P}^{n-1}$ under the action of the Heisenberg group.*

5.2 Abstract Configurations

Again here, we largely follow [Hul86]. In this section we restrict to the case where $n = p \geq 3$ is a prime number. Recall that there are exactly $p+1$ subgroups $\mathbb{Z}_p \subset \mathbb{Z}_p \times \mathbb{Z}_p$, that are generated by $(0,1)$ and $(1,l)$, $l \in \mathbb{Z}_p$ respectively. To describe the configurations acted upon the the Heisenberg group, we need to determine all hyperplane $H \subset \mathbb{P}^{p-1}$ which are invariant under one of these subgroups.

We start with the subgroup generated by $(0, 1)$. Clearly

$$\tau(H) = H \quad \text{if and only if} \quad H_k = \{x_{-k} = 0\}.$$

Further we note that

$$H_k = \sigma^k(H_0).$$

Now we will determine all hyperplane H such that

$$\tau^l \sigma(H) = H.$$

First of all, due to σ, the equation of any such H must be of the form

$$x_0 + \sum_{m=1}^{p-1} \lambda_m x_m = 0.$$

Then invariance under $\tau^l \sigma$ is equivalent to

$$\begin{aligned} \lambda_1^p &= 1, \\ \lambda_m &= \lambda_1^m \cdot \epsilon^{\frac{1}{2}m(m-l)}, \quad \text{for } m = 2, \ldots, p-1. \end{aligned}$$

Hence in setting

$$\lambda_1 = \epsilon^{-\frac{1}{2}(p-1)l-k}$$

for some $k \in \mathbb{Z}_p$, the other λ_m's become

$$\lambda_m = \epsilon^{\frac{m}{2}(m-p)l-mk}.$$

It follows that the p hyperplanes

$$H_{kl} = \left\{ \sum_{m=0}^{p-1} \epsilon^{\frac{m}{2}(m-p)l-mk} x_m = 0 : k = 0, \ldots, p-1 \right\}$$

are exactly the hyperplanes invariant under $\tau^l \sigma$. Also we observe that

$$H_{kl} = \tau^k(H_{0l}).$$

To summarise:

Proposition 5.3 ([Hul86]). *For each of the $p+1$ subgroups $\mathbb{Z}_p \subset \mathbb{Z}_p \times \mathbb{Z}_p$ there are exactly p hyperplanes which are invariant under this subgroup.*

Now we look at the involution $\imath(x_m) = x_{-m}$ on $V = \mathbb{C}^p$. It defines a decomposition of V

into eigenspaces, namely

$$\mathbb{C}^p = E^+ \oplus E^-,$$

where

$$E^+ = \left\langle e_0,\ e_1 + e_{p-1},\ \dots\ ,\ e_{\frac{p-1}{2}} + e_{\frac{p+1}{2}} \right\rangle,$$
$$E^- = \left\langle e_1 - e_{p-1},\ \dots\ ,\ e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}} \right\rangle.$$

Clearly $\dim E^+ = \frac{1}{2}(p+1)$ and $\dim E^- = \frac{1}{2}(p-1)$.

Lemma 5.4. *[Hul86]* $E^- = H_0 \cap H_{0,0} \cap \ldots \cap H_{0,p-1}$.

Proof. 1. We shall first show that E^- is contained in this intersection. Clearly $E^- \subseteq H_0$. Furthermore since H_{0l} is given by

$$\sum_{m=0}^{p-1} \lambda_m^l x_m = 0, \qquad \text{where} \qquad \lambda_m^l = \epsilon^{\frac{1}{2}m(m-p)l}.$$

The assertion now follows from

$$\lambda_{p-m}^l = \epsilon^{\frac{1}{2}(p-m)(-m)} = \epsilon^{\frac{1}{2}m(m-p)l} = \lambda_m^l.$$

2. To finish the proof, we need to show that the $\frac{1}{2}(p+1)$ of the hypersurfaces H_0 and H_{0l} are independent. This boils down to examining the matrix

$$V = \begin{bmatrix} 1 & 1 & \lambda_0 & \lambda_0^2 & \dots & \lambda_0^{p-1} \\ 0 & 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 1 & \lambda_{p-1} & \lambda_{p-1}^2 & \dots & \lambda_{p-1}^{p-1} \end{bmatrix}$$

Using the formula for the Vandermonde determinant, namely

$$\det(V) = \prod_{0 \le i < j \le p-1} (\lambda_j - \lambda_i),$$

it suffices to see that $\frac{1}{2}(p+1)$ of the λ_m's are different. Hence we look at

$$\lambda_{2k} = \epsilon^{k(2k-p)} = \epsilon^{2k^2}.$$

It suffices to see that $2k^2 \not\equiv 2l^2 \mod p$ for $k \neq l \in \{0, \dots, \frac{p-1}{2}\}$. But this is clearly so since othersince p would divide $2(k-l)(l+k)$, which is impossible.

□

We now define for all $k, l \in \mathbb{Z}_p$ the subspaces

$$E_{kl} := \tau^k \sigma^l (E^-).$$

Lemma 5.5. *$E_{kl} \cap E_{k'l'} = 0$ if $(k,l) \neq (k',l')$.*

Proof. We will prove that

$$E_{00} \cap E_{-k,-l} = 0$$

for $(k,l) \neq (0,0)$. To see this, assume that

$$x = \sum_{m=0}^{p-1} x_m e_m \in E_{00} \cap E_{-k,-l}. \tag{39}$$

Then, since $x \in E_{00}$ it follows that $x_m = -x_{-m}$. On the other hand, since $x \in E_{-k,-l}$, it follows that $\tau^k \sigma^l (x) \in E_{00}$. This is equivalent to

$$x_{m+l} \epsilon^{2mk} = -x_{-m+l}. \tag{40}$$

If $l = 0$ and $k \neq 0$, it follows from (39) and 40 that $x = 0$. Hence assume $l \neq 0$. by (39) it follows that $x_0 = 0$. Setting $m = -l$ in (40) implies $x_{2l} = 0$, which because of (39) leads to $x_{-2l} = 0$. Using again (40), this time with $m = -3l$ we find $x_{4l} = 0$. Continuing in this fashion one finds that $x = 0$. □

We want to sum up the situation as follows: we have found $p(p+1)$ hyperplanes which we denoted by H_k and H_{kl} respectively. Moreover we have constructed p^2 subspaces E_{kl} of dimension $\frac{1}{2}(p-1)$. Each of the spaces E_{kl} is contained in exactly $p+1$ of the hyperplanes and is in fact their common intersection. On the other hand, each of the hyperplanes H_k and H_{kl} contains exactly p of the subspaces E_{kl} and is indeed spanned by any two of them. In particular, we can say:

Proposition 5.6 ([Hul86])**.** *The $p(p+1)$ hyperplanes H_k and H_{kl} together with the p^2 subspaces E_{kl} form a configuration of type $(p^2_{p+1}, p(p+1)_p)$.*

We have not said anything in this section about the relation of this configuration to the embedded elliptic curve E_τ in $\mathbb{P}^{p-1}$. Because of

$$x_m(0) = -x_{-m}(0)$$

it follows that the (projective) space $E_{00} = E^-$ contains the origin O. Hence each of the subspaces E_{kl} goes through exactly one of the p-torsion points E_τ, namely $P_{kl} = \frac{k+l\tau}{p}$. In fact, this is the only point of intersection of E_{kl} with E_τ.

Since the hyperplanes H_k and H_{kl} are invariant under some subgroup $\mathbb{Z}_p \subset E_\tau[p]$, it follows that they all contain exactly p of the p-torsion points. On the other hand the hyperplanes are determined by these points. The exact relation is given by

$$H_k \ni \{mP_{01} + kP_{10} : m \in \mathbb{Z}_p\} = \left\{\frac{k + m\tau}{p} : m \in \mathbb{Z}_p\right\},$$
$$H_{kl} \ni \{mP_{1l} + kP_{01} : m \in \mathbb{Z}_p\} = \left\{\frac{m(1 + l\tau)}{p} : m \in \mathbb{Z}_p\right\}.$$

We summarise this as follows:

Proposition 5.7. *[Hul86] Each of the hyperplanes H_k and H_{kl} intersect E_τ in exactly p of the p-torsion points. The union of all p hyperplanes belonging to a fixed subgroup $\mathbb{Z}_p \subset E_\tau[p]$ contains all p^2 hyperosculating points of E_τ.*

In Section 5.4.1, we shall use the results of this section to scrutinise the configuration that the singular members of the Hesse pencil give rise to.

5.3 The Normaliser of the Heisenberg Group

We largely follow [Nie92] in this section, and begin by citing a theorem:

Theorem 5.8 (Discrete Theorem of Mumford-Stone-von Neumann, [Mum66])**.** *The Schrödinger representation of H_n is the unique irreducible representation of H_n such that the centre operates by natural scalar multiplication.*

The normaliser N_n of the Heisenberg group H_n is defined to be

$$N_n = \{n \in \mathrm{SL}(n, \mathbb{C}) : n \cdot H_n \cdot n^{-1} = H_n\}.$$

Conjugation defines a group homomorphism

$$N_n \to \mathrm{Aut}(H_n) \to \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}),$$

where

$$\begin{aligned} \mathrm{Aut}(H_n) &\longrightarrow \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}) \\ f &\longmapsto \bar{f} \end{aligned}$$

and $\bar{f}$ is the induced group automorphism to the quotient H_n/μ_n, which is well-defined since $f(\mu_n) = \mu_n$ by Theorem 5.8, and must belong to $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ since the automorphism preserves the bilinear form (38).

If $n \in H_n$, then the inner automorphism $t \mapsto n \cdot t \cdot n^{-1}$ of H_n induces the identity of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = H_n/\mu_n$. Conversely we claim that:

Claim 5.9 ([Nie92]). *For any element $n \in N_n$ such that conjugation with this element induces the identity on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, then n belong to H_n.*

Proof. By assumption, $n \cdot z \cdot n^{-1} = c \cdot z$ with $c \in \mu_n$. The scalar c depends only on the class $\bar{z}$ of z in $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, and $\bar{z} \mapsto c$ defines a homomorphism $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mu_n$. The form (,) from (38) on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is non-degenerate, so there is some $z_n \in H_n$ such that

$$c(\bar{z}) = \epsilon^{(\bar{z}_n, \bar{z})}$$

for all $z \in H_n$. This shows that

$$z_n \cdot z \cdot z_n^{-1} = c(\bar{z}) \cdot z = n \cdot z \cdot n^{-1}$$

for all z. Schur's Lemma then implies that n differs from z_n by a scalar factor in μ_n, hence $n \in H_n$. □

This assertion proves that the following sequence is exact:

$$1 \longrightarrow H_n \longrightarrow N_n \longrightarrow \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}).$$

We will come back to proving that the last map is surjective in our examples for $n = 3$ and 5, as it has a very interesting geometric interpretation that we would like to emphasise in its own right.

5.4 Examples

We now want to apply the results of the previous subsection to those of our elliptic curves E_τ found in Section 4. We begin with when $n = p = 3$ with our Hesse pencil.

5.4.1 The Hesse Pencil

An in-depth summary of the nature of the Hesse pencil is [AD06], which we borrow heavily from in this section. We recall that the Hesse pencil is given by the family of elliptic normal curves of degree 3, embedded in $\mathbb{P}^2$ by the homogeneous cubic polynomial

$$E_\lambda : x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2 = 0, \quad \lambda \in \mathbb{P}^1,$$

which has four singular members precisely when $\lambda \in \{1, \epsilon, \epsilon^2, \infty\}$, see Table 1. There are

Table 1: The inflection lines and the inflection points in the Hesse configuration.

Singular curves	Subgroup	Inflection lines		Inflection Points
E_∞	$\langle\tau\rangle$	H_0	$x_0 = 0$	E_{00}, E_{10}, E_{20}
		H_2	$x_1 = 0$	E_{02}, E_{12}, E_{22}
		H_1	$x_2 = 0$	E_{01}, E_{11}, E_{21}
E_1	$\langle\sigma\rangle$	H_{00}	$x_0 + x_1 + x_2 = 0$	E_{00}, E_{01}, E_{02}
		H_{10}	$x_0 + \epsilon x_1 + \epsilon^2 x_2 = 0$	E_{20}, E_{21}, E_{22}
		H_{20}	$x_0 + \epsilon^2 x_1 + \epsilon x_2 = 0$	E_{10}, E_{11}, E_{12}
E_{ϵ^2}	$\langle\tau\sigma\rangle$	H_{01}	$x_0 + \epsilon x_1 + \epsilon x_2 = 0$	E_{00}, E_{12}, E_{21}
		H_{11}	$x_0 + \epsilon^2 x_1 + x_2 = 0$	E_{02}, E_{20}, E_{22}
		H_{21}	$x_0 + x_1 + \epsilon^2 x_2 = 0$	E_{01}, E_{10}, E_{22}
E_ϵ	$\langle\tau^2\sigma\rangle$	H_{02}	$x_0 + \epsilon^2 x_1 + \epsilon^2 x_2 = 0$	E_{00}, E_{11}, E_{22}
		H_{12}	$x_0 + x_1 + \epsilon x_2 = 0$	E_{01}, E_{12}, E_{20}
		H_{22}	$x_0 + \epsilon x_1 + x_2 = 0$	E_{02}, E_{10}, E_{21}

also 9 base-points to the pencil which coincide with the 9 inflection point that make up the 3-torsion subgroup $E_\lambda[3]$[4]. They are:

$$\begin{aligned} E_{00} &= (0:1:-1), \quad E_{01} = (1:-1:0), \quad E_{02} = (1:0:-1), \\ E_{10} &= (0:1:-\epsilon), \quad E_{11} = (1:-\epsilon:0), \quad E_{12} = (1:0:-\epsilon^2), \\ E_{20} &= (0:1:-\epsilon^2), \quad E_{21} = (1:-\epsilon^2:0), \quad E_{22} = (1:0:-\epsilon). \end{aligned} \tag{41}$$

There are twelve singular points of each triangles, which we call the *vertices of the triangles*, which are:

$$\begin{aligned} v_1^{(10)} &= (1:0:0), \quad v_2^{(10)} = (0:1:0), \quad v_3^{(10)} = (0:0:1), \\ v_1^{(01)} &= (1:1:1), \quad v_2^{(01)} = (1:\epsilon:\epsilon^2), \quad v_3^{(01)} = (1:\epsilon^2:\epsilon), \\ v_1^{(11)} &= (\epsilon:1:1), \quad v_2^{(11)} = (1:\epsilon:1), \quad v_3^{(11)} = (1:1:\epsilon), \\ v_1^{(21)} &= (\epsilon^2:1:1), \quad v_2^{(21)} = (1:\epsilon^2:1), \quad v_3^{(21)} = (1:1:\epsilon^2). \end{aligned} \tag{42}$$

Each subgroup of the Heisenberg group H_3 has three vertices in (42) as fixed points, namely

$$\begin{aligned} \text{fix}(\langle\sigma\rangle) &= \{v_1^{(01)}, v_2^{(01)}, v_2^{(01)}\}, \qquad \text{fix}(\langle\tau\rangle) = \{v_1^{(10)}, v_2^{(10)}, v_3^{(10)}\}, \\ \text{fix}(\langle\tau\sigma\rangle) &= \{v_1^{(11)}, v_2^{(11)}, v_2^{(11)}\}, \qquad \text{fix}(\langle\tau^2\sigma\rangle) = \{v_1^{(21)}, v_2^{(21)}, v_3^{(21)}\}, \end{aligned}$$

with each set constituting a degenerate orbit of a given cyclic subgroup of order three under the action of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ in $\mathbb{P}^2$. The Hesse pencil gives rise to a $(9_4, 12_3)$ configuration,

[4]It is well known that three points on an elliptic curve add up to zero, if and only if they lie on an inflection line. Since the Hesse pencil is comprised of the Fermat cubic and a term proportional to its Hessian, this is no surprise.

known as the "Wendepunkts configuration", [Hul86].

We continue our discussion of the normaliser N_3 for the Heisenberg group H_3, by showing that the map

$$N_3 \to \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})$$

is surjective. To this end, consider the two transformations, originally considered by Bianchi [Bia80],

$$\delta = \frac{1}{\epsilon^2 - \epsilon} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{pmatrix}, \quad \nu = \frac{1}{\epsilon^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \tag{43}$$

They act via conjugation via

$$\begin{aligned} \delta \cdot \sigma \cdot \delta^{-1} &= \tau^2, & \delta \cdot \tau \cdot \delta^{-1} &= \sigma, \\ \nu \cdot \sigma \cdot \nu^{-1} &= \sigma\tau^2, & \nu \cdot \tau \cdot \nu^{-1} &= \tau = \tau, \end{aligned}$$

so indeed $\delta, \nu \in N_3$ and their images in $\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})$ are

$$\bar{\delta} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \bar{\nu} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

and these two matrices generate $\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})$. Hence the sequence

$$1 \longrightarrow H_3 \longrightarrow N_3 \longrightarrow \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}) \longrightarrow 1$$

is exact, and the normaliser N_3 is equal to the semi-direct product [HM73]

$$N_3 = H_3 \rtimes \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}),$$

where $\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})$ acts by its natural linear action on H_3, and its order is $|N_3| = 27 \cdot 24 = 648$. Let us investigate the action of both δ and ν on the singular members of the Hesse pencil. Substituting them in, we find that their action on the projective parameter $\lambda \in \mathbb{P}^1$ is the following:

$$\begin{aligned} \bar{\delta}(\lambda) &= \epsilon^2 \lambda, \\ \bar{\nu}(\lambda) &= \frac{\lambda + 2}{\lambda - 1}, \end{aligned}$$

and in particular:

$$\begin{aligned} \bar{\delta}(1) = \epsilon^2, \quad \bar{\delta}(\epsilon) = 1, \quad \bar{\delta}(\epsilon^2) = \epsilon, \quad \bar{\delta}(\infty) = \infty, \\ \bar{\nu}(1) = \infty, \quad \bar{\nu}(\epsilon) = \epsilon^2, \quad \bar{\nu}(\epsilon^2) = \epsilon, \quad \bar{\nu}(\infty) = 1. \end{aligned}$$

Now topologically, $\mathbb{P}^1$ is equivalent to the 2-sphere, S^2. So, by stereographically projecting

$\mathbb{P}^1$ onto S^2, we can view the four singular values for λ of the Hesse pencil as inscribing the vertices of a tetrahedron. Then the projective action of $\bar{\delta}$ and $\bar{\nu}$ permutes the vertices. Indeed, this follows from the well-known fact that $\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})/\mu_3 \cong \mathrm{PSL}(2, \mathbb{Z}/3\mathbb{Z}) \cong A_4$. Moreover, when mapped to $\mathrm{PGL}(3, \mathbb{C})$ we see that

$$N_3/\mathbb{C}^* \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}),$$

which has order 216. It is therefore isomorphic to the Hessian group G_{216} of order 216, since they both have the same order and both are the groups that preserve the Hesse pencil [AD06].

We finish this section with an interesting connection between the Hesse pencil and Shioda's modular surface $S(3)$[5], in which we follow [AD06]: consider the rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1, \quad (x_0 : x_1 : x_2) \mapsto (x_0x_1x_2 : x_0^3 + x_1^3 + x_2^3),$$

which is not defined at the nine base points of the pencil. Let

$$\pi : S(3) \longrightarrow \mathbb{P}^2$$

denote the blow up of the base points. This is a rational map such that the composition of rational maps $S(3) \to \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is a regular map

$$\phi : S(3) \longrightarrow \mathbb{P}^1,$$

whose fibres are isomorphic to the members of the Hesse pencil. In fact, the map ϕ defines a structure of a minimal elliptic surface on $S(3)$, called *Shioda's modular surface of level 3*, whose nine sections that define the 3-torsion subgroup in each fibre come from the exceptional curves that arise from the blowing up process, and the action of $\Gamma = \langle \sigma, \tau \rangle$ on $\mathbb{P}^2$ lifts to $S(3)$, and be identified with the Mordell-Weil group of the elliptic surface and its action with the translation action [AD06].

Let $\bar{\phi} : S(3)/\Gamma \to \mathbb{P}^1$ be the induced morphism from ϕ.

Proposition 5.10 ([AD06]). *The quotient surface $S(3)/\Gamma$ has four singular points of type A_2, or equivalently of type I_3 in Kodaira's classification*[6] *of singular fibres [Kod63] given by the singular orbits of the vertices (42). The minimal resolution of the singularities are isomorphic to $S(3)$, and up to these resolutions, $\bar{\phi}$ is isomorphic to ϕ.*

Proof. The group Γ preserves each singular member of the Hesse pencil and any of its subgroups leaves invariant the vertices of one of the triangles. Without loss of generality,

[5]For references on Shioda's modular surfaces $S(n)$, see [BH85]

[6]In Kodaira's classification, a singular fibre of type I_n is a connected cycle of n-cycles, each cycle being isomorphic to $\mathbb{P}^1$. In this case each cycle is one of the hyperplanes H_k and H_{kl}, $k, l \in \mathbb{Z}/3\mathbb{Z}$.

assume that the triangle is $x_0x_1x_2 = 0$. The the subgroup of Γ stabilising its vertices is $\langle\tau\rangle$, with acts locally at the point $x_1 = x_2 = 0$ by

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{C}^2 \\ (x_0 : x_1 : x_2) &\longmapsto \left(u = \frac{x_1}{x_0}, v = \frac{x_2}{x_0}\right), \end{aligned}$$

and hence the local action of τ is $\tau \cdot (u, v) = (\epsilon u, \epsilon^2 v)$. Hence the vertices give four singular points of type A_2 in $S(3)/\Gamma$, locally given by the equation $xy + z^3 = 0$.

The multiplication-by-3 map, $[3] : E \to E : \ x \mapsto 3x$ is a surjective map of degree 3^2, with kernel $E[3]$ by Proposition 2.14. The quotient map by Γ acts of each member of the Hesse pencil as the map [3], which implies that the quotient of the surface $S(3)$ by the group Γ is isomorphic to $S(3)$ over the open subset $U = \mathbb{P}^1 \setminus \{\infty, 1, \epsilon, \epsilon^2\}$. The map $\bar{\phi} : S(3)/\Gamma$ induced by the map ϕ has four singular fibres. Each fibre is an irreducible rational curve with a double point of type A_2. Let $\pi : S(3)' \to S(3)/\Gamma$ be a minimal resolution of the four singular points of $S(3)/\Gamma$. Then the composition $\bar{\phi} \circ \pi : S(3)' \to \mathbb{P}^1$ is an elliptic surface isomorphic to $\phi : S(3) \to \mathbb{P}^1$ over the open subset of the base $\mathbb{P}^1$. Moreover $\bar{\phi} \circ \pi$ and ϕ have singular fibres of the same types, thus $S(3)'$ is a minimal elliptic surface. As it is known that a birational isomorphism of minimal elliptic surfaces is an isomorphism [AD06], this implies that $\bar{\phi} \circ \pi$ is isomorphic to ϕ. $\square$

5.4.2 The Bianchi Quintic

We now turn our attention to the $n = 5$ case, where E_τ is cut out the the set-theoretic intersection of the five quadrics

$$Q_i(x_0, \ldots, x_4) = x_i^2 + a x_{i+2} x_{i+3} + \tfrac{1}{a} x_{i+1} x_{i+4}, \qquad i \in \mathbb{Z}/5\mathbb{Z},$$

where $a \in \mathbb{P}^1$ is a parameter depending solely on τ. Let

$$S_a = \bigcap_{i \in \mathbb{Z}/5\mathbb{Z}} Q_i(x_0, \ldots, x_4)$$

be the variety in $\mathbb{P}^4$ by the Q_i. We quote the following result from [BH85]:

Proposition 5.11. *For each $a \in \mathbb{P}^1$, S_a is a curve in $\mathbb{P}^4$. If $a \in \mathbb{P}^1 \setminus \Gamma$, where*

$$\Gamma = \{0, \infty, -(1/2)(1 \pm 5)\epsilon^k : k = 0, \ldots 4\} \subset \mathbb{P}^1,$$

where $\epsilon = \exp(2\pi i/5)$, the curve S_a is a smooth elliptic curve. On the other hand, if $a \in \Gamma$, then S_a is a connected cycle of five lines which is defined as type I_5 in Kodaira's classification of singular fibres [Kod63]. The twelve points in Γ can be identified with the *twelve vertices of an isosahedron inscribed within $S^2 \cong \mathbb{P}^1$.*

Analogously to our Hesse pencil, we want to derive a similar result to the action of the Heisenberg group on the twelve singular points for $a \in \Gamma$. To this end we will show the homomorphism

$$N_5 \longrightarrow \mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$$

is surjective, so that the normaliser of N_5 of H_5 in $\mathrm{SL}(5, \mathbb{C})$ is the semi-direct product

$$N_5 = H_5 \rtimes \mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z}).$$

To show this, we again define two matrices $\delta, \nu \in \mathrm{SL}(5, \mathbb{C})$, given by[7]

$$\delta = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 \\ 1 & \epsilon^2 & \epsilon^4 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^4 & \epsilon^3 & \epsilon^3 & \epsilon \end{pmatrix}, \qquad \nu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & \epsilon^4 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^4 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix},$$

so that

$$\begin{aligned} \delta \cdot \sigma \cdot \delta^{-1} &= \tau^4, & \delta \cdot \tau \cdot \delta^{-1} &= \sigma, \\ \nu \cdot \sigma \cdot \nu^{-1} &= \sigma\tau^2, & \nu \cdot \tau \cdot \nu^{-1} &= \tau = \tau. \end{aligned}$$

Therefore their images in $\mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$ are

$$\bar{\delta} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \quad \bar{\nu} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

and these generate $\mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$. So the map $N_5 \to \mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$ is surjective and N_5 can be written as the semi-direct product

$$N_5 = H_5 \rtimes \mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z}).$$

Its order is $|N_5| = 125 \cdot 120 = 15000$, and this group is the famous automorphism group of the Horrocks-Mumford bundle [HM73], whose order is listed in the title of their article. We however focus on the central quotient

$$\mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})/\mu_5 \cong \mathrm{PSL}(2, \mathbb{Z}/5\mathbb{Z}),$$

since $\mathrm{PSL}(2, \mathbb{Z}/5\mathbb{Z}) \cong A_5$, the *icosahedral group* [KM07]. Thus its action on $\mathbb{P}^1 \cong S^2$ can be viewed analogously to that of the tetrahedral group A_4 in the previous section, namely that its action on $\mathbb{P}^1$ can be identified with the permutations between the twelve singular

[7]Used both in [HM73] and [Bia80].

points in Γ.

6 Conclusion

We have used the basis vector space of theta functions of weight n to embed elliptic curves as different models in projective space $\mathbb{P}^{n-1}$, and found that when $n = 3, 4, 5$, the model is the Hesse pencil, the Fermat quartic, and the Bianchi quintic respectively. Moreover, we have provided the general formula for the models in higher dimensional projective space.

Furthermore the multiple symmetries of the embedded curves can be described through the action of the Heisenberg group, which acts as a representation group for the subgroup of n-torsion points of the curve. Even more interesting is the action of the normaliser of the Heisenberg group on the curve, which acts as a group of automorphisms on the family when $n = 3, 5$, and whose action of the singular members can be identified with the action of the tetrahedral and icosahedral groups on the 2-sphere, respectively.

The first natural direction for further study would be to investigate the nature of the quadric intersections of (26) for larger values of n. Indeed when $n = 7$, there are 14 quadrics from (26), and Gross shows in [GP96] that the coefficient matrix of the Pfaffians of (31) gives rise to Klein's quartic

$$K = \{(y_1 : y_2 : y_3) \in \mathbb{P}^2 : y_1^3 y_2 + y_2^3 y_3 + y_3^3 y_1 = 0\},$$

which is also the unique $\mathrm{PSL}(2, \mathbb{Z}/7\mathbb{Z})$ invariant of degree ≤ 4, and is therefore the isomorphic image in $\mathbb{P}^6$ of the modular curve $X(7)$. A similar example arises when $n = 11$, when an analogous cases arise for the modular curve $X(11)$, [GP96].

It would also be interesting to investigate more thoroughly the action of the Heisenberg group on the elliptic fibrations that the curves give rise to in a similar vein to Proposition (5.10), and their relation to Shioda's modular surfaces $S(n)$ as has been done in [BH85].

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