Contents

1 Recap & Motivation

Recall from Ana's talk, the Atiyah-Guillemin-Sternberg convexity theorem [1, 2].

Theorem 1.1 (Atiyah, Guillemin-Sternberg). Let (M, ω) be a compact connected symplectic manifold, and let T^n be an n-torus. Consider moment map $\mu : M \to \mathbb{R}^n = \mathrm{Lie}(T^n)$ for hamiltonian T^n-action.

- the level sets $\mu^{-1}(c)$ are connected $\forall c \in \mathbb{R}^n$;
- the image $\mu(M)$ is convex;
- the image $\mu(M)$ is the convex hull of the images of the fixed points of the action.

The image $\mu(M)$ of the moment map is called the *moment polytope*.

Example 1. Consider $\mathbb{CP}^2$ with the Fubini-Study symplectic form $(\mathbb{CP}^2, \omega_{FS})$. The T^2-action on $\mathbb{CP}^2$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 e^{i\theta_1} : z_2 e^{i\theta_2}]$$

has moment map

$$\mu([z_0 : z_1 : z_2]) = -\frac{1}{2}\left(\frac{|z_1|^2}{\|z\|^2}, \frac{|z_2|^2}{\|z\|^2}\right).$$

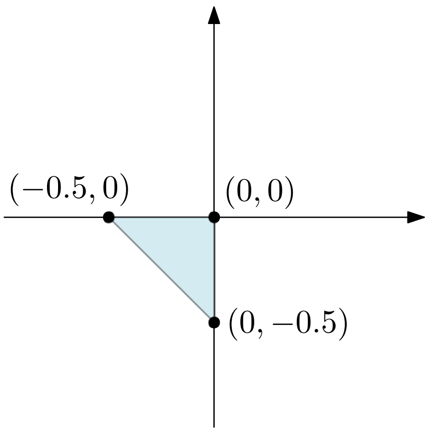
There are three fixed points of the T^2-action, namely

$$[1 : 0 : 0], \qquad [0 : 1 : 0], \qquad [0 : 0 : 1],$$

which get mapped to

$$(0, 0), \qquad (-\tfrac{1}{2}, 0), \qquad (0, -\tfrac{1}{2}),$$

respectively. Hence the moment polytope $\mu(\mathbb{CP}^2)$ is the right-angled triangle:



Question 2. Is this reversible? That is, given a convex polytope $\Delta \subset \mathbb{R}^n$, does there exist a compact connected symplectic manifold $(M_\Delta, \omega_\Delta)$ with some Lie group action $G \circlearrowright M_\Delta$, such that $\mu_\Delta(M_\Delta) = \Delta$? If so, is M_Δ unique?

A partial converse to the AGS convexity theorem is given by the Delzant construction.

2 Symplectic Toric Manifolds & the Delzant Construction

Definition 3. A *symplectic toric manifold* is a compact connected symplectic manifold (M^{2n}, ω) with an effective Hamiltonian action of a torus T^n of dimension equal to half the dimension of the manifold

$$\dim T^n = \frac{1}{2} \dim M^{2n},$$

and with a choice of corresponding moment map $\mu : M \to \mathbb{R}^n$.

The previous example is symplectic toric manifold:
$M = \mathbb{CP}^2$, and $G = S^1 \times S^1 = T^1$, so $\dim_{\mathbb{C}} T^1 = \frac{1}{2} \dim_{\mathbb{C}} \mathbb{CP}^2$.

Definition 4. A *Delzant polytope* Δ in $\mathbb{R}^n$ is a polytope satisfying:

simplicity: there are n edges meeting at each vertex;

rationality: each edge that meets a vertex p is of the form $p + tu_i$, with $t_i \geq 0$ and $u_i \in \mathbb{Z}^n$;

smoothness: for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

It turns out that the moment polytope of a symplectic toric manifold is Delzant.

Lemma 2.1. For any symplectic toric manifold (M, ω), its moment polytope is Delzant.

Proof. Let $x \in M$ be a fixed point of the action, then $\mu(x) = p$ is a vertex of $\Delta = \mu(M)$. In the proof of the AGS convecity theorem, one proves that there exists a neighbourhood of x that locally looks like the convex cone

$$\left\{p + \sum_{i=1}^{n} t_t u_i : t_i \geq 0,\ i = 1, \ldots n\right\},$$

where the $u_1, \ldots u_n$ are the weights of the linearised T^n-action on T_xM, say $\pi : T^n \to GL(T_xM)$. So Δ satisfies the simplicity and rationality conditions.

Now suppose that Δ does not satisfy the smoothness condition. Then the matrix $U = [u_1\ u_2\ \ldots\ u_n]$ is not invertible as a $\mathbb{Z}$-matrix. Pick some $a \in U^{-1}(\mathbb{Z}^n)$ such that $a \in \mathbb{R}^n - \mathbb{Z}^n$, then (using the bi-invariant metric on $\mathfrak{t}$) we get that $\langle w_i, a\rangle \in \mathbb{Z}$ for each $i = 1, \ldots, n$. But as the u_i are the weights of the action, then (where $\pi \cong \oplus_{i=1}^n \pi_i$, with each π_i of degree one because T is abelian) in the neighbourhood

$$\pi_i(\exp(a)) \cdot z = e^{2\pi i \langle u_i, a\rangle} z = z \qquad \text{for } i = 1, \ldots n,$$

for all points z in the convex neighbourhood of x. But $\exp(a)$ cannot be the identity element since in a dense open subset of M the action is free. □

So this shows that any toric symplectic manifold has, as the image of its moment map, a Delzant polytope associated to it.

Theorem 2.2 (Delzant, [3]). Toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:

$$\frac{\{\text{toric manifolds}\}}{\{T^n\text{-equivariant symplectomorphisms}\}} \longleftrightarrow \frac{\{\text{Delzant polytopes}\}}{\{\text{translations}\}}$$
$$(M_\Delta^{2n}, \omega_\Delta, T^n, \mu) \longleftrightarrow \mu(M_\Delta) = \Delta.$$

Sketch of proof. One can break the proof down into several steps:

1. M is toric $\implies \mu(M)$ is Delzant (already done).
2. Δ is Delzant $\implies$ there exists a compact connected symplectic manifold $(M_\Delta, \omega_\Delta)$.
3. Show that M_Δ is toric with $\mu(M_\Delta) = \Delta$.
4. Show that if $M_1 \simeq M_2 \implies \mu(M_1) = \mu(M_2)$.
5. Show that if $\mu(M_1) = \mu(M_2) \implies M_1 \simeq M_2$.

We will prove the surjectivity statement (Step 2.) in the theorem, so that to any Delzant polytope Δ there exists a corresponding toric symplectic manifold M_Δ. Then we can mimick the steps with a particular example for some Delzant polytope Δ or some toric symplectic manifold (M, ω) with a hamiltonian torus action $T \circlearrowright M$, and see either what toric symplectic M or Delzant polytope Δ pops out, respectively.

Let $\pi : \mathbb{R}^d \to \mathbb{R}^n$ be the linear map $\pi : e_i \mapsto v_i$. The map is surjective as Δ is Delzant, and maps $\mathbb{Z}^d \to \mathbb{Z}^n$, so induces a map between tori (still denoted by π)

$$\pi : \mathbb{R}^d/2\pi\mathbb{Z}^d \longrightarrow \mathbb{R}^n/2\pi\mathbb{Z}^n,$$

whose kernel $N = \ker(\pi)$ is an $(n-d)$-dimensional Lie subgroup of T^d, with the inclusion $i : N \hookrightarrow T^d$. The exact sequence

$$1 \longrightarrow N \xrightarrow{\ i\ } T^d \xrightarrow{\ \pi\ } T^n \longrightarrow 1$$

induces the exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\ i_*\ } \mathbb{R}^d \xrightarrow{\ \pi_*\ } \mathbb{R}^n \longrightarrow 0,$$

and corresponding dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\ \pi^*\ } (\mathbb{R}^d)^* \xrightarrow{\ i^*\ } \mathfrak{n} \longrightarrow 0.$$

Now let $\mathbb{C}^d$ be acted upon by the diagonal Hamiltonian action of T^d

$$e^{i\theta_j} \cdot z_j = e^{i\theta_j} z$$

which has the moment map $\phi : \mathbb{C}^d \to (\mathbb{R}^d)^*$

$$\phi(z_1, \ldots, z_d) = -\frac{1}{2}(|z_1|^2, \ldots |z_d|^2) + \text{ constant},$$

where we choose the constant to be $(\lambda_1, \ldots, \lambda_d)$. Restricting the action on $\mathbb{C}^d$ to the subgroup N, it remains Hamiltonian [4] with moment map

$$\mu_N := i^* \circ \phi : \mathbb{C}^d \to \mathfrak{n}^*.$$

Set $Z := \mu_N^{-1}(0)$ to be the zero-level set; it can be shown that Z is compact symplectic and N acts freely on Z (see [4]). Since i^* is surjective, $0 \in \mathfrak{n}^*$ is a regular value of μ_N, and so $Z \subset \mathbb{C}^d$ is a compact submanifold of real dimension

$$\dim_{\mathbb{R}} Z = \dim_{\mathbb{R}} \mathbb{C}^d - \dim_{\mathbb{R}} \mathfrak{n}^* = 2d - (d-n) = d+n.$$

By the Marsden-Weinstein-Meyer theorem, the orbit space $M_\Delta = Z/N$ is also a compact manifold, and its dimension is

$$\dim M_\Delta = \dim_{\mathbb{R}} Z - \dim_{\mathbb{R}} N = (n+d) - (d-n) = 2n.$$

The point-orbit map $p : Z \to Z/N = M_\Delta$ is a principal N-bundle over M_Δ. By considering the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ j\ } & \mathbb{C}^d \\ \big\downarrow{\scriptstyle p} & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is the inclusion, and let ω_0 be the standard symplectic form on $\mathbb{C}^d$. Then the Marsden-Weinstein-Meyer theorem guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^*\omega_\Delta = j^*\omega_0.$$

Finally, as Z is connected, the compact symplectic $2n$-dimensional manifold $(M_\Delta, \omega_\Delta)$ is connected [4] □

Remark 5. Step 4 is actually straightforward to prove: if $M_1 \simeq M_2$, then the different moment maps for the same T^n-action differ only by a constant in the dual Lie algebra. Thus the corresponding moment polytopes differ by a translation.

Since the symplectic form ω_Δ on M_Δ coincides with that of ω_0 on $\mathbb{C}^d$ when both are pulled back to Z, we have the following:

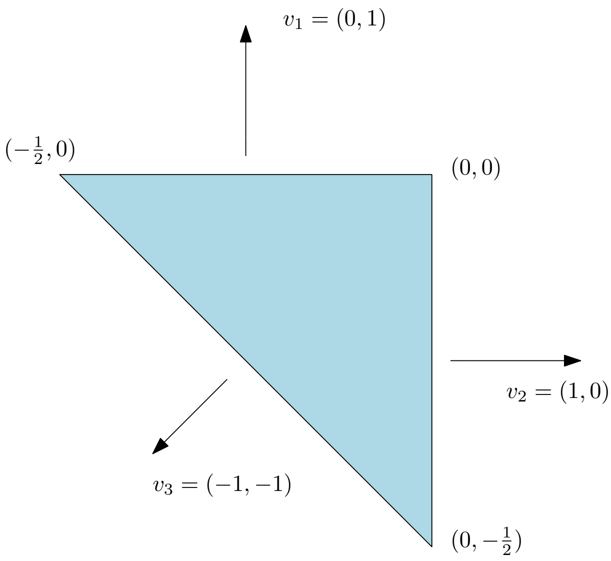
Corollary 6. *The symplectic manifold $(M_\Delta, \omega_\Delta)$ constructed above has a natural Kähler structure.*

Remark 7. Let Δ be a Delzant polyope in $(\mathbb{R}^n)^*$ and with d facets. Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots d$, be the primitive outward-pointing normal vectors to the facets of Δ. Then Δ can be described as n intersection of half-spaces

$$\Delta = \{x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \ldots, d\} \quad \text{for some } \lambda_i \in \mathbb{R}.$$

Example 8. From the $T^2 \circlearrowright \mathbb{CP}^2$ example from before:

$$\begin{aligned} \Delta &= \left\{x \in (\mathbb{R}^2)^* : x_1 \leq 0,\ x_2 \leq 0,\ x_1 + x_2 \geq -\tfrac{1}{2}\right\} \\ &= \left\{x \in (\mathbb{R}^2)^* : \langle x, (1,0) \rangle \leq 0,\ \langle x, (0,1) \rangle \leq 0,\ \langle x, (-1,-1) \rangle \leq \tfrac{1}{2}\right\} \end{aligned}$$



3 Examples

3.1 $\mathbb{CP}^1$ with a T^1-action

Consider $\Delta = [-1, 1] \subset \mathbb{R}^*$, so $n = 1$ and $d = 2$. Let v $(= 1)$ be the standard basis vector in $\mathbb{R}$. Then

$$\Delta = \{x \in \mathbb{R}^* : \langle x, 1\rangle \leq 1,\ \langle x, -1\rangle \leq 1\}.$$

The projection

$$\pi_* : \mathbb{R}^2 \to \mathbb{R} : \quad (1, 0) \mapsto 1,\ (0, 1) \mapsto -1$$

has kernel $\ker \pi_* = \{(t, t) : t \in \mathbb{R}\}$, hence $\ker \pi = \{(e^{it}, e^{it}) : t \in \mathbb{R}\} \cong S^1 = N$. Now $i : N \hookrightarrow T^2$ is given by the inclusion $e^{it} \mapsto (e^{it}, e^{it})$, and so $i_*(t) = (t, t)$. From this we can determine the structure of $i^* : (\mathbb{R}^2)^* \to \mathfrak{n}$ using the standard inner-product:

$$\begin{aligned}\langle i^*(x, y), z\rangle &= \langle (x, y), i_*(z)\rangle \\ &= \langle (x, y), (z, z)\rangle = (x + y)z = \langle (x + y), z\rangle,\end{aligned}$$

so $i^*(x, y) = x + y \in \mathfrak{n}^*$. The standard diagonal action of T^2 on $\mathbb{C}^2$ has the moment map

$$\phi(z_1, z_2) = \left(-\frac{|z_1|^2}{2}, -\frac{|z_2|^2}{2}\right) + (\lambda_1, \lambda_2),$$

where we will set $\lambda_1 = \lambda_2 = 1$. The induced moment map $\mu_\Delta = i^* \circ \phi : \mathbb{C}^2 \to \mathfrak{n}^* \cong \mathbb{R}$ is then

$$\begin{aligned}i^* \circ \phi(z_1, z_2) &= i^*\left(1 - \frac{|z_1|^2}{2}, 1 - \frac{|z_2|^2}{2}\right) \\ &= 2 - \left(\frac{|z_1|^2}{2} + \frac{|z_2|^2}{2}\right),\end{aligned}$$

whence

$$Z = (i^* \circ \phi)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 4\} \cong S^3.$$

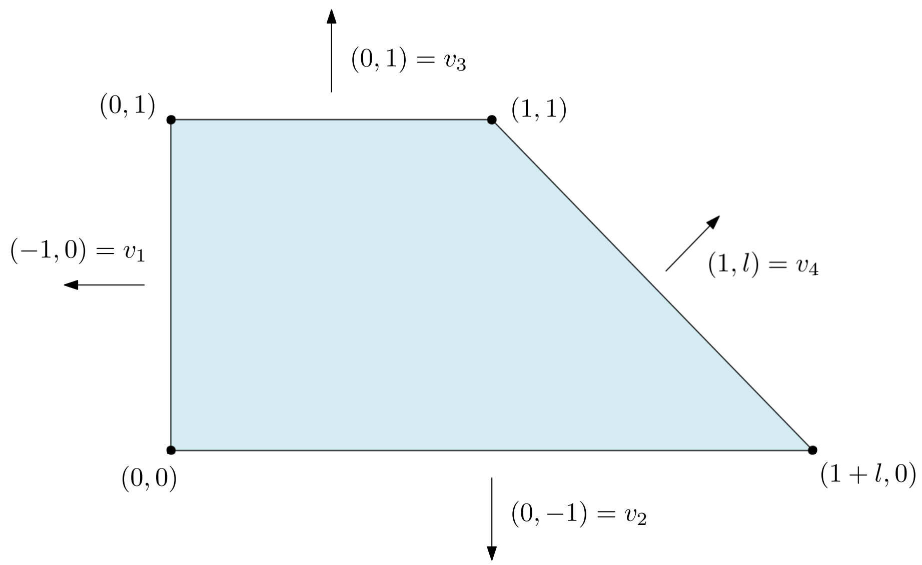
Finally, we quotient out by the action of $N \cong S^1$ i.e. by the equivalence $(z_1, z_2) \sim (e^{it}z_1, e^{it}z_2)$ to get that $M_\Delta = Z/N \cong S^3/S^1 \cong \mathbb{CP}^1$.

Remark 9. It should not be hard to see that $M_\Delta \cong \mathbb{CP}^1$ for all 1-dimensional Delzant polytopes Δ. In general, the toric action of T^n on $\mathbb{CP}^n$ has an n-dimensional polytope $\Delta \subset \mathbb{R}^n$ formed from the convex hull of n-fixed points of the action, which are the $n+1$ points $[\dots, 0, 1, 0, \dots] \in \mathbb{CP}^n$.

3.2 T^2-action on $\mathbb{C}^4$

Consider the polytope

$$\begin{aligned}\Delta &= \big\{(x_1, x_2) \in (\mathbb{R}^2)^* : x_1 \geq 0,\ 0 \leq x_2 \leq 1,\ x_1 + lx_2 \leq 1 + l\big\} \\ &= \big\{(x_1, x_2) \in (\mathbb{R}^2)^* : \langle x, (-1, 0)\rangle \leq 0,\ \langle x, (0, -1)\rangle \leq 0,\ \langle x, (0, 1)\rangle \leq 1,\ \langle x, (1, l)\rangle \leq 1 + l\big\} \\ &= \bigcap_{i=1}^{4} \{(x_1, x_2) \in (\mathbb{R}^2)^* : \langle x, v_i\rangle \leq \lambda_i\}\end{aligned}$$



We have 4 edges (facets) so $d = 4$, and the polytope is 2-dimensional so $n = 2$. From the second equality for Δ above, we see that $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, 1, 1 + l)$. Since

$$\pi_* := \mathbb{R}^4 \to \mathbb{R}^2 : e_i \mapsto v_i,$$

π_* can be written down as a 2×4 matrix as $\pi_* = [v_1 \ \dots \ v_4]$, namely

$$\pi_* = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & l \end{pmatrix}$$

hence

$$\pi_*(A, B, C, D) = (-A + D, -B + C + lD).$$

The corresponding group element of T^4 is mapped to an element of T^2 as

$$\pi(a, b, c, d) = (a^{-1}d, b^{-1}cd^l),$$

hence the kernel of π is

$$N = \ker(\pi) = \{(a, b, c, d) : a = d, \ b = cd^l\} \cong T^2.$$

Now the inclusion map $i : N \hookrightarrow T^4$ is

$$(a, b) \longmapsto (a, b, ba^{-l}, a),$$

and its push-forward between Lie algebras $i_* : \mathfrak{n} \to \mathbb{R}^4$ is

$$(A, B) \longmapsto (A, B, B - lA, A).$$

We need to find the form of the pull-back $i^* : (\mathbb{R}^4)^* \to \mathfrak{n}^*$ in order to find $\mu_\Delta = i^* \circ \phi$, which we can determine using the standard inner product on $\mathbb{R}^2$; for $X = (A, B) \in \mathfrak{n}$ and $Y = (C_1, C_2, C_3, C_4) \in \mathfrak{n}^*$,

$$\begin{aligned} \langle Y, X \rangle &= \Big\langle i^*(C_1, \ C_2, C_3, C_4), \ (A, B) \Big\rangle \\ &= \Big\langle (C_1, C_2, C_3, C_4), \ i_*(A, B) \Big\rangle \\ &= \Big\langle (C_1, C_2, C_3, C_4), \ (A, B, B - lA, A) \Big\rangle \\ &= (C_1 - lC_3 + C_4)A + (C_2 + C_3)B \\ &= \Big\langle (C_1 - lC_3 + C_4, C_2 + C_3), \ (A, B) \Big\rangle, \end{aligned}$$

from which one deduces that

$$i^*(C_1, C_2, C_3, C_4) = (C_1 - lC_3 + C_4, \ C_2 + C_3).$$

Next, we consider the moment map

$$\begin{aligned}\phi : \mathbb{C}^4 &\longrightarrow (\mathbb{R}^4)^*\\ (z_1, z_2, z_3, z_4) &\longmapsto -\frac{1}{2}\big(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2\big) + (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\\ &= \frac{1}{2}\Big(- |z_1|^2, -|z_2|^2, -|z_3|^2 + 2, -|z_4|^2 + 2l + 2\Big)\end{aligned}$$

Now we restrict the hamiltonian action on $\mathbb{C}^d$ to the subtorus N with moment map

$$\begin{aligned}\mu_\Delta = i^* \circ \phi : \mathbb{C}^d &\longrightarrow \mathfrak{n}^*\\ (z_1, z_2, z_3, z_4) &\longmapsto \frac{1}{2}\Big(- |z_1|^2 + l|z_3|^2 - |z_4|^2 + 2,\ -|z_2|^2 - |z_3|^2 + 2\Big).\end{aligned}$$

The zero-level set $Z = \mu_N^{-1}(0)$ can now be seen to be determined by

$$|z_1|^2 - l|z_3|^2 + |z_4|^2 = 2, \qquad \text{and} \qquad |z_2|^2 + |z_3|^2 = 2,$$

so neither of the pairs (z_1, z_4) and (z_2, z_3) can equal zero. Finally, $\dim_\mathbb{R} Z = 6$ and so after taking the quotient of Z by the action of $N = T^2$, we get the toric symplectic manifold $M_\Delta = Z/N$ of dimension $\dim_\mathbb{R} M_\Delta = 4$.

Claim 10. *M_Δ is the l^{th} Hirzebuch surface.*

Proof. I will relabel the coordinates as $(z_1, z_2, z_3, z_4) = (x_1, y_1, y_2, x_2)$ and l as p, so that Z is now determined by the equations

$$\begin{aligned}|x_1|^2 - p|y_2|^2 + |x_2|^2 &= 2,\\ |y_1|^2 + |y_2|^2 &= 2,\end{aligned}$$

and where $N = S^1 \times S^1$ acts on Z as

$$\begin{aligned}(\alpha, 1) \cdot (x_1, x_2; y_1, y_2) &= (\alpha x_1, \alpha x_2; y_1, \alpha^{-p} y_2),\\ (1, \beta) \cdot (x_1, x_2; y_1, y_2) &= (x_1, x_2; \beta y_1, \beta y_2).\end{aligned}$$

The ratio $[x_1 : x_2]$ is preserved by the action of N, so projecting to the first factors gives us a diffeomorphism $p : M_\Delta \to \mathbb{CP}^1$. The fibre above any given ratio $[x_1 : x_2] \in \mathbb{CP}^1$ has to be invariant under the action of N. The fibre above a point (x_0, x_1) is invariant under

$$\alpha \cdot \big((x_1, x_2), [y_1 : y_2]\big) = \big((\alpha x_1, \alpha x_2), [0 : y_1 : \alpha^{-p} y_2]\big),$$

where $[1 : 0 : 0]$ is the point at infinity in each fibre, disjoint to the affine patches. When

$U_1 = \{x_1 \neq 0\}$ and $U_2 = \{x_2 \neq 0\}$, it can be covered affinely by

$$\begin{aligned} &x_2/x_1, \qquad x_1^p y_2/y_1, \qquad \text{on } U_1, \\ &x_1/x_2, \qquad x_2^p y_2/y_1, \qquad \text{on } U_2, \end{aligned}$$

so the transition function $g_{12} : U_1 \cap U_2 \to GL(2, \mathbb{C}^*)$ is $g_{12}(x_1, x_2) = \text{diag}(1, (x_2/x_1)^p)$, which is the cocycle that determines the sum $\mathcal{O}_{\mathbb{CP}} \oplus \mathcal{O}_{\mathbb{CP}}(-p)$ bundle. Projectivising (i.e. adding in $[1 : 0 : 0]$ to each fibre) shows that M_Δ is the total space of $\mathbb{P}(\mathcal{O}_{\mathbb{CP}} \oplus \mathcal{O}_{\mathbb{CP}}(-p))$, known as the p^{th} Hirzebuch surface. □

Other interpretations of M_Δ for each $p \geq 1$ is as the blow-up of the weighted projective space $\mathbb{P}(1, 1, p)$ at a point (see [5]), or as a rational normal scroll $\mathbb{F}(p, 0)$ (see [6], chapter 2). Note that the Delzant polytope here can be obtained from the one for $\mathbb{CP}^2$ by cutting off one of the vertices (in a prescribed way). This operation is equivalent to a symplectic blow-up (see [4]).

References

[1] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Inventiones mathematicae*, 67(3):491–513, 1982.

[2] M. F. Atiyah. Convexity and commuting hamiltonians. *Bulletin of the London Mathematical Society*, 14(1):1–15, 1982.

[3] Thomas Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bulletin de la Société Mathématique de France*, 116(3):315–339, 1988.

[4] Ana Cannas Da Silva and A Cannas Da Salva. *Lectures on symplectic geometry*, volume 3575. Springer, 2001.

[5] Paul Gauduchon. *Hirzebruch Surfaces and Weighted Projective Planes*, pages 25–48. 07 2010.

[6] Miles Reid. Chapters on algebraic surfaces. *arXiv e-prints*, pages alg–geom/9602006, Feb 1996.